PHD

Large Deviations for Boundary Driven Exclusion Processes

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Abstract

We study the totally asymmetric exclusion process on the positive integers with a single particle source at the origin. Liggett (1975) has shown that the long term behaviour of this process has a phase transition: If the particle production rate at the source and the initial density are below certain critical values, the stationary measure is a product measure, otherwise the stationary measure is spatially correlated. Following the approach of Derrida *et al.* (1993) it was shown by Großkinsky (2004) that these correlations can be described by means of a matrix product representation. In this thesis we derive a large deviation principle with explicit rate function for the particle density in a macroscopic box based on this representation. The novel and rigorous technique we develop for this problem combines spectral theoretical and combinatorial ideas and has the potential to be applicable to other models described by matrix products.

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"If I have seen further it is by standing on the shoulders of giants."

Sir Isaac Newton

To my mum and dad.

To my siblings, Gabs and José. For science and world peace. . . RAWR!

I really want to thank Amine: To changing the world. . . Cheers!

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CONTENTS

CHAPTER 1

INTRODUCTION

Many natural systems are not in thermodynamic equilibrium, which loosely speaking means that there is a permanent exchange of energy or matter of the system with its surroundings or within the system itself. In statistical physics, the simple exclusion process is often considered the paradigm of such a system out of equilibrium. In the absence of a general theory for systems out of equilibrium, it has been argued that large deviation rate functions play an important role as a replacement for the thermodynamical potential [5]. The principal aim of this thesis is to develop a rigorous mathematical technique to derive such rate functions from a particular type of representation of the stationary state of the system, the matrix products, which twenty years after the pioneering work of Derrida *et al.* [14] is available for a wide range of particle systems out of equilibrium, see for example Blythe and Evans [7] for a survey.

The method is presented in the case of the totally asymmetric exclusion process (TASEP) on the positive integers with a single particle source at the origin, a case which has apparently not been treated in the literature so far. In this Markovian model, particles are positioned on the sites of the semi-infinite lattice $\mathbb{N} = \{1, 2, \dots\}$

in such a way that no site carries more than one particle. The dynamics of the model can be informally described as follows: A particle source carries a Poisson clock with intensity $\alpha > 0$. If this clock rings, the source attempts to inject a particle at site one. If this site is vacant the injection takes place, otherwise it is suppressed and nothing happens. Also, every particle in the system carries an independent Poisson clock with rate one, and when the clock rings the particle tries to jump to the neighbouring site on its right. If this site is vacant the jump takes place, otherwise it is suppressed. Note that the exclusion interaction originating from the suppression of jumps and injections ensures that no site ever carries more than one particle.

The exclusion interaction in this model has a profound effect on the behaviour of the system. Most notably, the detailed balance equations for this Markov chain have no nontrivial solution. Hence the system is not reversible, in other words it is out of equilibrium. The long term behaviour of the process shows local convergence to a stationary measure which depends on the initial configuration of the system. If the injection rate α satisfies $\alpha \leq \frac{1}{2}$ and the initial density ρ is low enough, the system does not feel the interaction and the stationary measure is the product measure with density α. If however $\alpha > \frac{1}{2}$ or $\rho > 1 - \alpha$, the exclusion of particles leads to spatial correlations in the stationary measure, which is no longer a product measure. In this case, the overall particle density at stationarity is the maximum of $1/2$ and the initial density ρ, independently of the injection rate α.

There have been considerable efforts to describe the long range correlations of the stationary measures and the microscopic transition kernels in the exclusion process explicitly. For instance, Sasamoto and Williams [25] and Tracy and Widom [27] derive explicit formulas from combinatorial identities, and Sasamoto [24] uses an ansatz based on orthogonal polynomials. A particularly successful approach to describe spatial correlations is the matrix product ansatz first suggested in 1993 by Derrida, Evans, Hakim and Pasquier [14] and refined and extended in a large number of papers, see [12, 17, 20] for a few further examples.

Large deviation principles have been derived for the hydrodynamical limits of a range of boundary driven exclusion processes by Bertini and coauthors [4, 6] and the method should be extendable to our case. In principle, large deviation principles for the particle density in a macroscopic box then follow from these results by contraction, see [9]. However, the optimisation in path space, which is required to get an explicit rate function, is often unwieldy and technical as Bahadoran's paper [2] readily testifies.

In the light of these difficulties it is a natural idea to try and derive large deviation principles directly from the matrix product ansatz. This plan was carried out by Derrida *et al.* [15] in the case of an asymmetric exclusion process on a finite number of sites. Key to their method is a saddle point argument, which allows to derive an additivity formula which compares the stationary measure on the interval with stationary measures on complementary subintervals. From this formula an explicit rate function for the particle density is derived. The paper [15] was a spectacular success, but we have not been able to implement this method in the case of a semi-infinite lattice. In a different development, Angeletti *et al.* [1] show that already for matrix product representations with finite matrices the large deviation principles that arise from this exhibit a rich phenomenology. Finite matrix representations have the advantage that they can be studied using the Perron-Frobenius theory, which is unavailable for infinite matrices. Physical examples, however, are almost always based on representations by infinite matrices.

Here we present a rigorous and novel approach to calculate large deviations for the macroscopic particle density in the semi-infinite totally asymmetric exclusion process. We use the matrix product representation as a starting point, and base the analysis on the Gärtner-Ellis theorem. To study the asymptotics of the cumulant generating function of the particle density, we use quite different approaches for the lower and upper bounds. The lower bound is based on the spectral theory of Toeplitz operators in a suitable weighted sequence space, while the upper bound directly exploits combinatorial identities coming directly from the matrix product ansatz. As our method is not too technical, we believe that it is very promising to deal with a wide range of other particle systems whose stationary measure can be described by a matrix product representation.

It cannot be left unsaid that the work of done to produce this thesis also brought to life a paper [18], unpublished at the time of the writing of this work, but definitely a good thing to mention.

This thesis is organised as follows. In Chapter 2 we introduce the simple exclusion process in several variants: symmetric, asymmetric, weakly asymmetric and totally asymmetric. We mention that the state space of the process may be finite or infinite. We also go through the previous results needed to state and interpret our main theorem.

Chapter 3 discusses the matrix product representation in the particular case of the semi-infinite totally asymmetric simple exclusion process. A huge proportion of the existing literature on this subject is not written in the familiar linear algebra notation, an effort to translate everything into an easy introduction with familiar notation for a new postgraduate student in mathematics was made making of this chapter a good starting point. We also describe here our approach to the large deviation problem.

The proof of the upper bound for the limit cumulant generating function is carried out in Chapter 4. The techniques involved here are very well known results of functional analysis in weighted ℓ^2 spaces and Toeplitz operators applied on these.

The lower bound is then derived in Chapter 5. The flavour of the proof is combinatorial in nature contrasting with the previous chapter. Surprisingly we conclude that the lower and upper bounds match and we end with the final details of the proof of our main result.

Finally, some concluding remarks are made in Chapter 6.

Before starting allow me to motivate you with the final words of the immortal speech given in Königsberg on the 8 September 1930 by one the most influential mathematicians of the last century, David Hilbert:

"We must know. We will know."

CHAPTER 2

THE SIMPLE EXCLUSION PROCESS

2.1 Background

The simple exclusion process is one of the simplest models of interacting particle systems. Having said that, it does not mean it is a trivial system and in fact there are several variants to it. For completeness we will review their definitions, explain their differences, and show some explicit examples. In the last section of this chapter we will review the process we will be focusing on, namely, the semi-infinite totally asymmetric simple exclusion process.

In general, we characterise a Markov process $\{\xi_t\}_{t\geq 0}$ by its state space Ω, a non-empty set where the random variables of our process take values; and an infinitesimal generator G, that is an operator defined on functions $f\colon \Omega \to \mathbb{R}$ that determines the time evolution of the process.

We are now going to see the particular cases of these concepts for some of the several variants of the simple exclusion process. For an introduction to Markov processes in discrete and continuous time see [23], for a formal treatment of general Markov processes see [16].

2.2 Different types of SEP

2.2.1 The SSEP

The *symmetric simple exclusion process*, or simply SSEP, can be described as follows: Consider a one dimensional finite *lattice* with $n \in \mathbb{N}$ *sites*. Let there be a particle in each site of a subset of the n sites (the subset can be empty or it can be all of the sites). Starting from that *configuration*, the dynamics work like this: Each particle in the system will wait independently an exponential time with mean 1 and with probability $\frac{1}{2}$ it will jump to the site on the right and with the same probability it will jump to the site on the left. However, if there is a particle at the site where one of the particles decides to move the jump is suppressed. Once a particle either jumps or suppresses its jump it starts all over again, see figure 2-1.

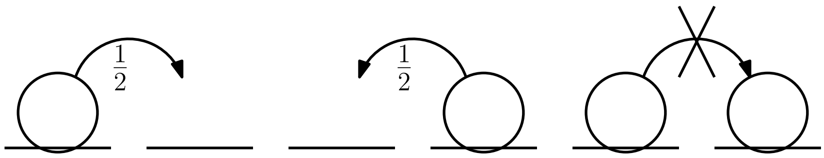

Figure 2-1: Symmetric simple exclusion process on a finite lattice.

There are some concepts that deserve a better description. By lattice we should understand the "space" where the particles move. In our first example, the SSEP on a finite lattice, we can think of the particles moving on the first n non-negative integers, that is the set $\{1, 2, \ldots, n\}$. But we may think of infinite lattices such as $\mathbb{N}$, $\mathbb{Z}$ or even $\mathbb{Z}^d$. However we will restrict ourselves to the finite set $\{1, 2, \ldots, n\}$ and what we will call the semi-infinite lattice $\mathbb{N}$. A site is an element of the lattice, naively a "place" where a particle can be. Finally, by configuration we mean a possible and valid arrangement of particles on the lattice, therefore we say η is a configuration if it is an element of the state space Ω.

To give a formal definition of the model, we first define the auxiliary switching and swapping functions $\sigma^x, \sigma^{x,y} \colon \Omega \to \Omega$ by

$$(\sigma^{x,y}\eta)_z = \begin{cases} \eta_y & \text{if } z = x \\ \eta_x & \text{if } z = y \\ \eta_z & \text{if } z \notin \{x, y\}, \end{cases} \qquad (\sigma^x\eta)_z = \begin{cases} 1 - \eta_x & \text{if } z = x \\ \eta_z & \text{if } z \neq x. \end{cases}$$

Formally, let $n \in \mathbb{N}$ and $\Omega = \{0,1\}^n$, a SSEP $\{\xi_t\}_{t \geq 0}$ is a Markov process with generator G such that for $f \colon \Omega \to \mathbb{R}$

$$\begin{aligned} Gf(\eta) = &\sum_{k=1}^{n-1} \eta_k(1 - \eta_{k+1}) \left(f(\sigma^{k,k+1}\eta) - f(\eta) \right) \\ &+ \sum_{k=2}^{n} \eta_k(1 - \eta_{k-1}) \left(f(\sigma^{k,k-1}\eta) - f(\eta) \right). \end{aligned}$$

2.2.2 The ASEP

It is easy to think of a more general process than the SSEP. The *asymmetric simple exclusion process*, or simply ASEP, is described by 5 parameters: a left injection rate $\alpha \in [0,1]$, a right extraction rate $\beta \in [0,1]$, a right injection rate $\gamma \in [0,1]$, a left extraction rate $\delta \in [0,1]$, and a left jump rate $q \in [0,1)$.

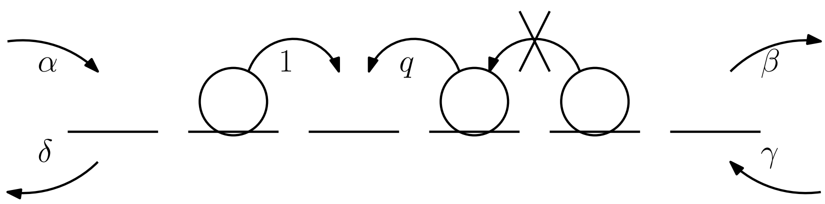

Figure 2-2: Asymmetric simple exclusion process on a finite lattice.

Note that now we are allowing particles to come in and go out of the system, see figure 2-2. We can allow this for the SSEP too although we would have to change the generator accordingly. For the case of this ASEP, the state space given the number of sites n is $\Omega = \{0,1\}^n$ and the generator is defined by

$$\begin{aligned} Gf(\eta) = &\sum_{k=2}^{n-1} \eta_k(1-\eta_{k+1}) \left(f(\sigma^{k,k+1}\eta) - f(\eta)\right) \\ &+ \sum_{k=2}^{n-1} q\eta_k(1-\eta_{k-1}) \left(f(\sigma^{k,k-1}\eta) - f(\eta)\right) \\ &+ \alpha(1-\eta_1)\left(f(\sigma^1\eta) - f(\eta)\right) + \beta\eta_n \left(f(\sigma^n\eta) - f(\eta)\right) \\ &+ \gamma(1-\eta_n)\left(f(\sigma^n\eta) - f(\eta)\right) + \delta\eta_1 \left(f(\sigma^1\eta) - f(\eta)\right). \end{aligned}$$

Note that if we allow $q = 1$ then the probability of a particle jumping to the left becomes $\frac{1}{2}$ since the rate of jumping to the right is also 1 as can be seen in the first line of the generator.

The simplest example of an ASEP where all parameters play a role is with $n = 3$, and the process in $\Omega = \{0,1\}^3$ can be seen as a random walk in the graph shown in figure 2-3.

It is an ergodic process and hence has a unique stationary measure. To find a stationary measure μ, we can find a non trivial solution to the system $\mu G = 0$ where G is the infinitesimal generator of the ASEP and it can be represented by a matrix as in 2.1:

$$G = \begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} \begin{pmatrix} g_1 & \gamma & 0 & 0 & \alpha & 0 & 0 & 0 \\ \beta & g_2 & q & 0 & 0 & \alpha & 0 & 0 \\ 0 & 1 & g_3 & \gamma & q & 0 & \alpha & 0 \\ 0 & 0 & \beta & g_4 & 0 & q & 0 & \alpha \\ \delta & 0 & 1 & 0 & g_5 & \gamma & 0 & 0 \\ 0 & \delta & 0 & 1 & \beta & g_6 & q & 0 \\ 0 & 0 & \delta & 0 & 0 & 1 & g_7 & \gamma \\ 0 & 0 & 0 & \delta & 0 & 0 & \beta & g_8 \end{pmatrix} \tag{2.1}$$

where the variables g_k are such that each row of G adds up to 0.

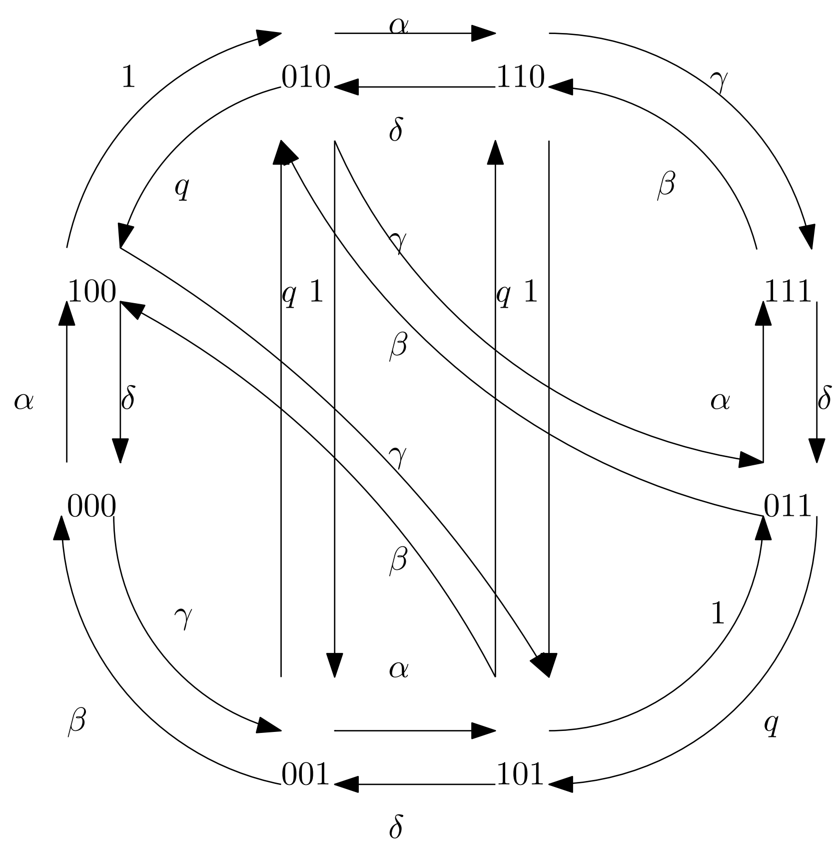

Figure 2-3: ASEP on 3 sites.

Although it seems there is a pattern for constructing these generator matrices for arbitrary values of n (see Proposition 2.1 below), it does not appear to be an easy way to solve the equation $\mu G = 0$ in general. We will see in the next chapter another way of finding the stationary measure.

2.2.3 The WASEP

Another variant is the *weakly asymmetric simple exclusion process.* Consider now a sequence of asymmetric processes, ASEPs were defined in the previous section, for which the parameters q depend on the number of sites and we let the asymmetry to disappear. That is, we have a sequence $\{q_n\}_{\mathbb{N}}$ of rates such that $q_n \to 1$ as $n \to \infty$.

Naively we would expect the limiting object should be a SSEP on an infinite lattice. We will use as an example the model from Bertini [6], the state space is $\Omega_N = \{0,1\}^{\{-N,-N+1,\dots,N-1,N\}}$ and the generator given parameters ρ_-, ρ_+, and E is defined by

$$\begin{aligned} G_N f(\eta) = {} & \tfrac{1}{2}\rho_- e^{E/2N}\left(f(\sigma^{-N+1}\eta) - f(\eta)\right) + \tfrac{1}{2}(1-\rho_-)e^{-E/2N}\left(f(\sigma^{-N+1}\eta) - f(\eta)\right) \\ & + \tfrac{1}{2}\rho_+ e^{-E/2N}\left(f(\sigma^{N-1}\eta) - f(\eta)\right) + \tfrac{1}{2}(1-\rho_+)e^{E/2N}\left(f(\sigma^{N-1}\eta) - f(\eta)\right) \\ & + \sum_{x=-N+1}^{N-1} \tfrac{1}{2}\exp\{-\tfrac{E}{2N}\left(\eta(x+1) - \eta(x)\right)\}\left(f(\sigma^{x,x+1}\eta) - f(\eta)\right). \end{aligned}$$

In this case, the rate of jumping to the left converges to $\frac{1}{2}$ instead of 1 as we previously stated. However, the rate of jumping to the right also converges to the same number which is not fixed at 1 in this case. Other versions can be seen in the works of Derrida and others [8, 13].

2.3 The TASEP

The *totally asymmetric simple exclusion process* is the model we will be focusing on this work, specifically on the semi-infinite TASEP, however we will start by introducing the TASEP in a finite lattice.

2.3.1 The finite TASEP

The finite TASEP on $n \in \mathbb{N}$ sites with injection rate $\alpha \in (0,1)$ and exit rate $\beta \in (0,1)$ has state space $\Omega = \{0,1\}^n$ and generator

$$\begin{aligned} Gf(\eta) = &\sum_{k=1}^{n-1} \eta_k(1-\eta_{k+1}) \Big(f(\sigma^{k,k+1}\eta) - f(\eta) \Big) \\ &+ \alpha(1-\eta_1) \big(f(\sigma^1\eta) - f(\eta) \big) + \beta\eta_n \left(f(\sigma^n\eta) - f(\eta) \right). \end{aligned}$$

Note that since it is finite and recurrent, it is an ergodic process and hence has a unique invariant distribution. In the next chapter we will see how this measure is found via the matrix product ansatz as seen on the work done by Derrida [14], however, we think it is worth mentioning here another matrix approach that has apparently not yet being exploited.

Proposition 2.1. *For a fixed number $n \in \mathbb{N}$ the infinitesimal generator of a finite TASEP with n sites, injection rate α, and exit rate β has a matrix representation $Q \in \mathbb{R}^{2^n \times 2^n}$ as follows, the non-diagonal elements, that is when $r \neq c$, by*

$$Q_{rc} = \begin{cases} \alpha & \text{if } r \in \{0,\dots,2^{n-1}-1\} \text{ and } c = r + 2^{n-1}, \\ \beta & \text{if } r \in \{1,3,\dots,2^n-3, 2^n-1\} \text{ and } c = r-1, \\ 1 & \text{if } r = 2^{n-k}(2j+1)+\ell \text{ and } c = r - 2^{n-k-1} \text{ for some triplet} \\ & \quad k \in \{1,\dots,n-1\}, j \in \{0,\dots,2^{k-1}-1\}, \ell \in \{0,\dots,2^{n-k-1}-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

And the diagonal elements are given by

$$Q_{rr} = -\sum_{c \neq r} Q_{rc}$$

We will prove this result not by induction as one might think but by counting the number of non-zero off diagonal elements of the infinitesimal generator matrix and assigning them the corresponding entry, exit or jump rate. After that we will just fill in the diagonal elements with the required values.

Proof. Let $n \in \mathbb{N}$ and $\{\xi(t)\}_{t\geq 0}$ a finite TASEP with injection rate α and exit rate β. Then the state space is $\Omega = \{0,1\}^n$ which is finite with cardinality 2^n, so we enumerate its elements $\Omega = \{\eta^0, \eta^1, \ldots, \eta^{2^n-1}\}$ where we identify k-th configuration with the binary expansion of the number k. For example, in figure 2-4 we can see a configuration and its corresponding binary expansion.

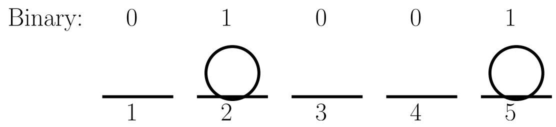

Figure 2-4: The binary expansion of number 9 and the corresponding 9-th configuration.

We are interested in constructing the matrix $Q \in \mathbb{R}^{2^n \times 2^n}$ for which the entry Q_{rc} is the rate of going from configuration η^r to configuration η^c. If $r = c$ the element $-Q_{rr}$ is the rate at which the process jumps out of configuration η^r.

Consider a pair of configurations η and ζ such that one can go from η to ζ simply by adding a particle to the first site. By the exclusion, this implies that η has no particle on the first site and $\zeta = \sigma^1(\eta)$, meaning that the binary expansion of η has a 0 as a first digit, $\eta_1 = 0$, and the expansion of ζ has a 1, $\zeta_1 = 1$. In the interpretation of the configuration as binary numbers, we have $\eta < 2^{n-1}$ and $\zeta = \eta + 2^{n-1} \geq 2^{n-1}$. Therefore $Q_{rc} = \alpha$ if and only if $r \in \{0, 1, \ldots, 2^{n-1} - 1\}$ and $c = r + 2^{n-1}$.

Consider now two configurations η and ζ such that we can go from η to ζ by taking out a particle from the last site. Then $\eta \equiv 1 \mod 2$ and $\zeta = \sigma^n(\eta) = \eta - 1$. Therefore $Q_{rc} = \beta$ if and only if $r \in \{1, 3, \ldots, 2^n - 1\}$ and $c = r - 1$.

Finally, consider configurations η and ζ such that one gets ζ from η by moving an already existing particle to the site on the right. Suppose this particle was at site k and moves now to site $k+1$, see figure 2-5.

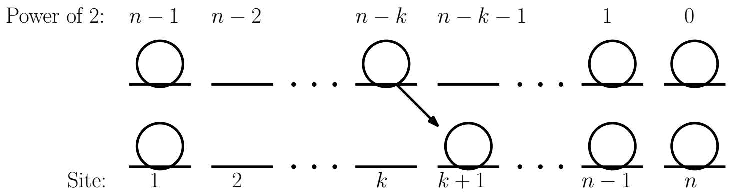

Figure 2-5: Movement of a pre-existing particle.

That means that

$$\zeta = \sigma^{k,k+1}(\eta) = \eta - 2^{n-k} + 2^{n-k-1} = \eta - 2^{n-k-1}.$$

Pick k to be any site from $\{1, 2, \ldots, n-1\}$ containing a particle, a movement of this type imposes site $k+1$ to be empty and the rest of the $n-2$ sites could be either empty or occupied. Hence, there are only $2^{n-2}(n-1)$ pairs of configurations for which $Q_{rc} = 1$. We claim these configurations are of the form $r = 2^{n-k}(2j+1) + \ell$ and $c = r - 2^{n-k-1}$ with some triplet $k \in \{1, \ldots, n-1\}$, $j \in \{0, \ldots, 2^{k-1}-1\}$, and $\ell \in \{0, \ldots, 2^{n-k-1}-1\}$. Note that these are exactly $2^{n-2}(n-1)$ pairs of configurations so we need to check that indeed r is a configuration with a particle at site k and empty at site $k+1$.

Fix the triplet (k, j, ℓ) and consider the binary expansion of j and ℓ

$$j = \sum_{p=0}^{k-2} j_p 2^p, \qquad \text{and} \qquad \ell = \sum_{p=0}^{n-k-2} \ell_p 2^p$$

then

$$\begin{aligned} r &= 2^{n-k}(2j+1) + \ell \\ &= 2^{n-k}\big(2\sum_{p=0}^{k-2} j_p 2^p + 1\big) + \sum_{p=0}^{n-k-2} \ell_p 2^p \\ &= \sum_{p=n-k+1}^{n-1} j_{p-n+k-1} 2^p + 2^{n-k} + \sum_{p=0}^{n-k-2} \ell_p 2^p. \end{aligned}$$

Since the coefficient for 2^{n-k} is 1, r has a particle at site k. The coefficient for 2^{n-k-1} is

0, hence r has no particle at site $k+1$. Since by construction $c = r - 2^{n-k-1} = \sigma^{k,k+1}(r)$, these are the only configurations such that $Q_{rc} = 1$.

The rest of the non-diagonal elements are 0 since it is impossible to get one configuration from another by simply moving one particle. The diagonal elements are calculated from the property $\sum_{c\in\Omega} Q_{rc} = 0$ for all $r \in \Omega$. ■

From Proposition 2.1, we can find the unique invariant probability measure of the finite TASEP as the vector satisfying the equations $\mu Q = 0$ and $\sum_{k\in\Omega} \mu_k = 1$. Nevertheless, we were not able to find an explicit solution for μ in vector form following this approach.

2.3.2 The semi-infinite TASEP

The first difference with respect to all the previous processes we have described is that the state space is not finite, in fact it is $\Omega = \{0,1\}^{\mathbb{N}}$ which is uncountably infinite. Let $\{\xi_t\}_{t\geq 0}$ be a semi-infinite TASEP with injection rate $\alpha \in (0,1)$ and semigroup $S(t)$ identified by its infinitesimal generator which depends only on the injection rate parameter $\alpha \in (0,1)$ and is defined on functions that depend only on a finite number of sites (cylindrical functions) by

$$\begin{aligned} Gf(\eta) = {}& \alpha(1-\eta_1)\left(f(\sigma^1\eta) - f(\eta)\right) \\ &+ \sum_{k\in\mathbb{N}} \eta_k(1-\eta_{k+1})\left(f(\sigma^{k,k+1}\eta) - f(\eta)\right). \end{aligned} \tag{2.2}$$

To continue the analysis of these processes first we need to introduce notation for two different sets of measures of Ω: We will denote by ν_α the *product measure* with constant density α, that is

$$\nu_\alpha\{\eta \in \{0,1\}^{\mathbb{N}} \colon \eta_{j_1} = 1, \eta_{j_2} = 1, \ldots, \eta_{j_n} = 1\} = \alpha^n$$

for all distinct choices of $j_1, j_2, \ldots, j_n \in \mathbb{N}$ and all $n \in \mathbb{N}$. And we say a measure μ_ρ is *asymptotically product* with density ρ if

$$\lim_{k\to\infty} \mu_\rho\{\eta \in \{0,1\}^{\mathbb{N}} \colon \eta_{j_1+k} = 1, \eta_{j_2+k} = 1, \ldots, \eta_{j_n+k} = 1\} = \rho^n$$

for all distinct choices of $j_1, j_2, \ldots, j_n \in \mathbb{N}$ and all $n \in \mathbb{N}$.

2.3.3 The stationary measures

Note that the semi-infinite TASEP is transient, since once the process visits a state it will never return to it. Because the process is not ergodic, it will not necessarily have a unique stationary measure. In fact, as proved in Theorem 1.8 of [22], we can find a set of stationary measures each of which may be reached depending on the initial conditions given by two parameters: the injection rate, which we will denote by α, and the initial measure, which we will only consider product measure with density ρ.

Theorem 2.2. *[22, Theorem 1.8] Let μ be a product measure on $\{0,1\}^{\mathbb{N}}$ for which $\rho := \lim_{k\to\infty} \mu\{\eta : \eta_k = 1\}$ exists. Then there exist probability measures μ_ϱ^α defined if*

- *either $\alpha \le \frac{1}{2}$ and $\varrho > 1 - \alpha$,*
- *or $\alpha > \frac{1}{2}$ and $\frac{1}{2} \le \varrho \le 1$,*

which are asymptotically product with density ϱ, such that

$$\text{if } \alpha \le \frac{1}{2} \text{ then } \quad \lim_{t\to\infty} \mu S(t) = \begin{cases} \nu_\alpha & \text{if } \rho \le 1 - \alpha \\ \mu_\rho^\alpha & \text{if } \rho > 1 - \alpha, \end{cases}$$

$$\text{and if } \alpha > \frac{1}{2} \text{ then } \quad \lim_{t\to\infty} \mu S(t) = \begin{cases} \mu_{1/2}^\alpha & \text{if } \rho \le \frac{1}{2} \\ \mu_\rho^\alpha & \text{if } \rho > \frac{1}{2}. \end{cases}$$

Theorem 2.2 states some cases when stationary measures exist but only characterises them up to an asymptotic density. Figure 2-6 summarises these asymptotic densities of the stationary measures as a function of the injection rate α and the initial density ρ. We will call the region bounded by $\alpha \leq \frac{1}{2}$ and $\rho \leq 1 - \alpha$ as non-interacting since it corresponds to the totally characterised stationary measures by product measures that only depend on the injection rate.

The region bounded by $\rho \geq \frac{1}{2}$ and $\rho \geq 1 - \alpha$ as bulk driven since the stationary measures depend only on the initial density. Finally, the region bounded by $\alpha \geq \frac{1}{2}$ and $\rho \leq \frac{1}{2}$ is the region of maximum current for reasons we will explore in the next chapter, note that the asymptotic density in this region is always $\frac{1}{2}$ regardless of the injection rate or the initial density. Note that in these two regions the invariant measures develop long range correlations. Observe too that convergence is not uniform on all sites, but that it depends on the initial condition, for example, starting from empty sites convergence slows down the further sites are from the origin.

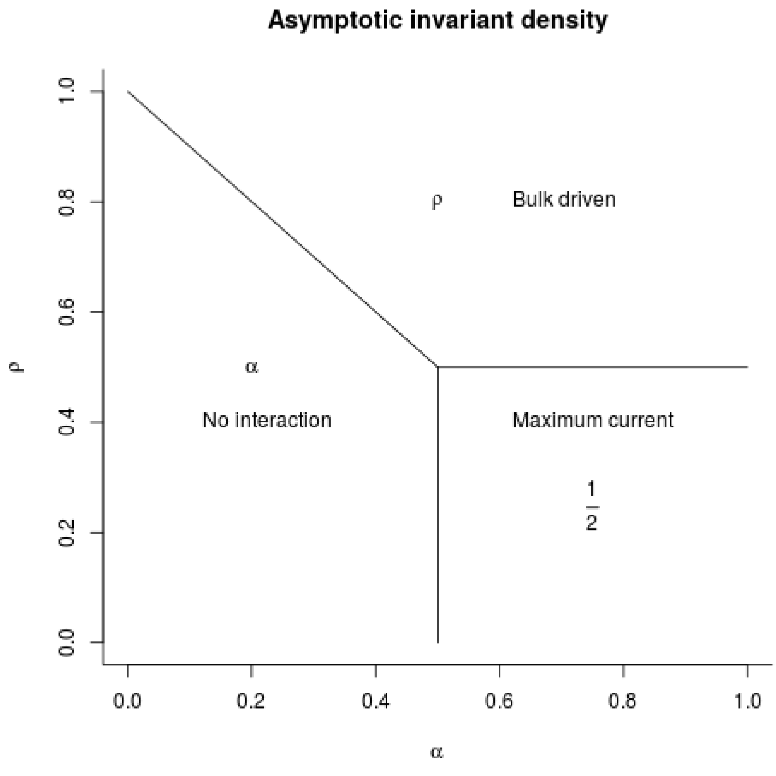

Figure 2-6: Asymptotic densities given by Theorem 2.2.

2.3.4 The empirical density

Even when Theorem 2.2 only tells us the behaviour of the stationary measure far from the origin, we will focus on the empirical density of the first n sites. For $n \in \mathbb{N}$ we define the empirical density as the random variable

$$X_n = \frac{1}{n} \sum_{k=1}^{n} \eta_k. \tag{2.3}$$

In principle, we may see this as a stochastic process when it depends on time. However, we will only focus on the stationary measures, meaning that we may get rid of the time dependence or, equivalently, we first take the limit as time goes to infinity.

2.4 The main result

The main result of this work is to find explicitly a rate function for a large deviation principle of the empirical density under the stationary measure given any injection rate α and initial density ρ in the region $0 \leq \rho < |\alpha - \frac{1}{2}| + \frac{1}{2}$.

2.4.1 Basics on the Theory of Large Deviations

The theory of large deviations is the analysis of the exponential decay rate of increasingly unlikely events. With our purpose in mind, we say that a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ taking values on $[0,1]$ satisfies a large deviation principle with rate function $I : [0,1] \to [0,\infty]$ under the probability measure $\mathbb{P}$ if

(i) the function I is lower semicontinuous,

(ii) for all open sets $G \subset [0,1]$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{X_n \in G\} \geq - \inf_{x\in G} I(x),$$

(iii) and for all closed sets $F \subset [0,1]$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{X_n \in F\} \le - \inf_{x\in F} I(x).$$

2.4.2 Main theorem

The following large deviation principle is the main result of this thesis.

Theorem 2.3. *Let $\{X_n\}_{n\in\mathbb{N}}$ be the sequence of random variables defined as the empirical density (2.3) of a semi-infinite TASEP with injection rate $\alpha \in (0,1)$ and initial asymptotically product measure for which ρ as defined in Theorem 2.2 exists. Then, under the stationary probability measure given by Theorem 2.2, $\{X_n\}_{n\in\mathbb{N}}$ satisfies a large deviation principle with convex rate function $I\colon [0,1] \to [0,\infty]$ given as follows.*

(a) If $\alpha \le \frac{1}{2}$ and $\rho < 1-\alpha$, then

$$I(x) = x\log\frac{x}{\alpha} + (1-x)\log\frac{1-x}{1-\alpha}.$$

(b) If $\alpha > \frac{1}{2}$ and $0 \le \rho \le \frac{1}{2}$, then

$$I(x) = \begin{cases} x\log\frac{x}{\alpha} + (1-x)\log\frac{1-x}{1-\alpha} + \log\left(4\alpha(1-\alpha)\right) & \textit{if} \quad 0 \le x \le 1-\alpha, \\ 2\left[x\log x + (1-x)\log(1-x) + \log 2\right] & \textit{if} \quad 1-\alpha < x \le \frac{1}{2}, \\ x\log x + (1-x)\log(1-x) + \log 2 & \textit{if} \quad \frac{1}{2} < x \le 1. \end{cases}$$

(c) If $\alpha > \frac{1}{2}$ and $\frac{1}{2} < \rho < \alpha$, then

$$I(x) = \begin{cases} x\log\frac{x}{\alpha} + (1-x)\log\frac{1-x}{1-\alpha} + \log\frac{\alpha(1-\alpha)}{\rho(1-\rho)} & \textit{if} \quad 0 \le x \le 1-\alpha, \\ 2\left[x\log x + (1-x)\log(1-x) - \log\sqrt{\rho(1-\rho)}\right] & \textit{if} \quad 1-\alpha < x \le 1-\rho, \\ x\log\frac{x}{\rho} + (1-x)\log\frac{1-x}{1-\rho} & \textit{if} \quad 1-\rho < x \le 1. \end{cases}$$

Figures 2-7, 2-8, 2-9 and 2-10 show plots for different values of α and ρ of the rate function given by Theorem 2.3. Note that part (a) of Theorem 2.3 corresponds to the rate function of the average of a sequence independent and identically distributed

Bernoulli random variables with parameter $\alpha \leq \frac{1}{2}$, this fact can be actually see from Theorem 2.2 already.

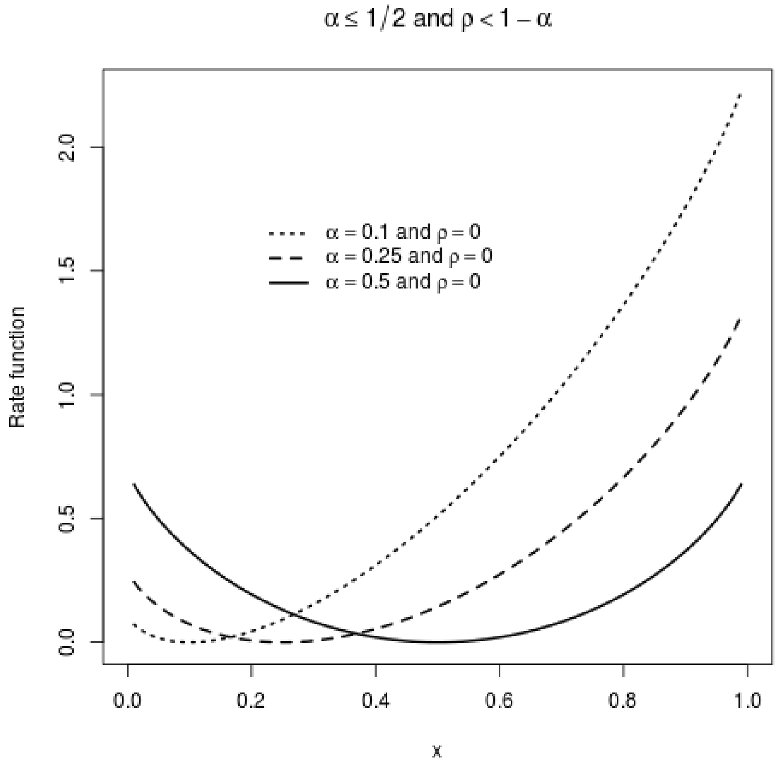

Figure 2-7: Rate functions for part (a)

Parts (b) and (c) of Theorem 2.3 are actually the new result. For part (b), in contrast with the previous case, the minimum of the rate function stays fixed at $x = \frac{1}{2}$ for all values of $\alpha > \frac{1}{2}$. High densities have a constant cost too that does not depend on α; however, low densities become increasingly expensive as α increases. Note there are two phase transitions in this case: at $x = \frac{1}{2}$ and another one at $x = 1-\alpha$. The first one may be naively explained as being a traffic overflow at the first site when particles are created: If the injection rate is too high there will be many cancelled particle creations due to the exclusion making the density stay at $\frac{1}{2}$ instead reaching the α value. Nevertheless, there is yet no intuitive explanation for the other phase transition.

For part (c), the minimum of the rate function is now at ρ. Low densities still become increasingly expensive as α increases; yet, high densities now become cheaper.

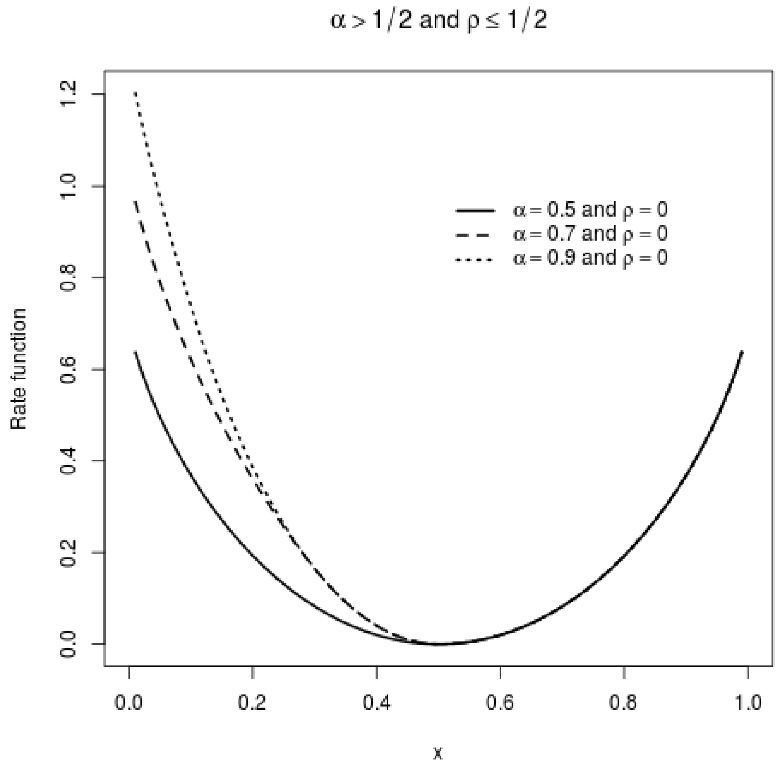

Figure 2-8: Rate functions for part (b)

In this case, we observe that both α and ρ play a role in the rate function, see figure 2-9 to appreciate the joint effect. The phase transitions are seen at $x = 1 - \alpha$, as in the previous case, and $x = 1 - \rho$.

It also worth comparing Theorem 2.3 with the rate function for the large deviation principle of the empirical density of the finite TASEP studied in [15]. For the case $\alpha \leq \frac{1}{2}$ and $\rho < 1 - \alpha$, we can choose an exit rate $\beta = 1 - \alpha$ that gives a stationary measure of independent Bernoulli random variables with parameter α, therefore in this case the finite TASEP and the finite box of the semi infinite TASEP have the same distribution. However, for the case $\alpha > \frac{1}{2}$ there is no $\beta \in [0, 1]$ that will give the same stationary measure as the one of the semi-infinite TASEP; nevertheless, if $\rho \leq \frac{1}{2}$ taking the exit rate $\beta = \frac{1}{2}$ gives us the same rate function, see [15, (3.12)]. It occurs the same taking $\beta = 1 - \rho$ if $\frac{1}{2} < \rho \leq \alpha$, we recover the same rate function as the finite TASEP, see again [15, (3.12)].

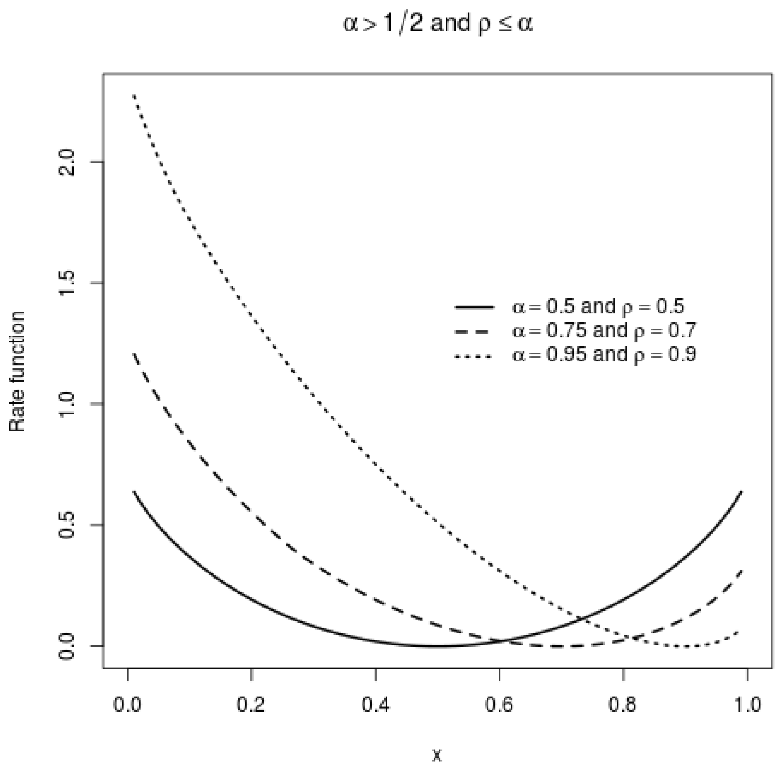

Figure 2-9: Rate functions for part (c) α and ρ increasing.

The rest of this work will set the necessary tools to prove Theorem 2.3. Recall that the whole parameter space of the semi-infinite TASEP is given in figure 2-6 and yet the theorem corresponds to only to three quarters of the parameter space. We will explain in due time why we were not able to extend it to the whole set so far.

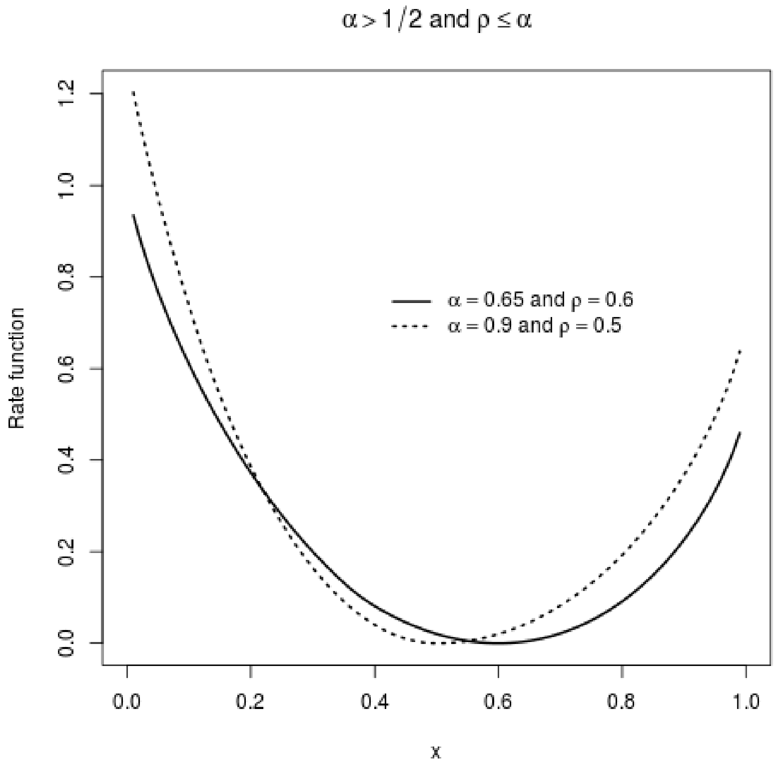

Figure 2-10: Rate functions for part (c) increasing α but decreasing ρ.

CHAPTER 3

MATRIX PRODUCT ANSATZ

The matrix product ansatz, or simply MPA, is a technique that consists in writing the invariant measure of an interacting particle system explicitly as the product of, generally non-commutative, matrices.

In this chapter, we will discuss the MPA for the semi-infinite TASEP only. We will be following the work of Großkinsky [19] but we should point out that the method was originally proposed by Derrida, Evans, Hakim and Pasquier [14].

Our strategy to find large deviation principles shown in this thesis seems very promising since the MPA method is not unique of the TASEP or the simple exclusion processes described in the previous chapter. The work of Klauck and Shadschneider [21] has shown that there is a broad class of interacting particle systems where the MPA has proven useful to describe explicitly the invariant measures.

The core of our strategy relies heavily on the MPA, hence we will start with the main result of this section and explain how it can be used to find a large deviation principle. At the end of this chapter we will show a failed attempt which shows a good example on the difference between working on finite dimensional spaces rather than directly on infinite dimensional ones.

3.1 The method

The idea is as follows, fix a configuration in a large box of n sites, say $\eta \in \{0,1\}^n$. We want to evaluate the stationary measure on the set of configurations of the state space whose sites on the first large box are given by configuration η. That is, we want to calculate the measure of the set $\{\zeta \in \{0,1\}^{\mathbb{N}} : \zeta_1 = \eta_1, \ldots, \zeta_n = \eta_n\}$. To do this we multiply a chain of matrices chosen from two, say D and E, in the following way: If site k of configuration η is empty we multiply by E; if however, the site has a particle we multiply by D. For example, if the configuration in the box of size 5 is $\eta = 01101$, as shown in figure 3-1, then the chain we need to multiply is $EDDED$. Since the result of multiplying these matrices is still a matrix we just multiply by the left and by the right by the 2 vectors, say w and v, and divide by the normalising constant to get a probability.

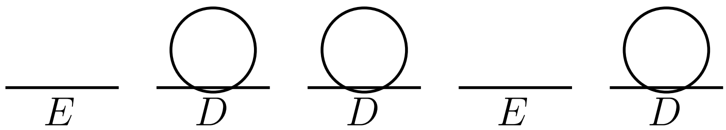

Figure 3-1: A configuration and its chain of matrices in the MPA.

That is roughly the idea, we state here the main result of this chapter, its proof is in [19].

Theorem 3.1. *[19, Theorem 3.2] Suppose there exist (possibly infinite) non-negative matrices D, E and vectors w and v, fulfilling the algebraic relations*

$$DE = D + E, \tag{3.1a}$$

$$\alpha w^T E = w^T, \tag{3.1b}$$

$$c(D + E)v = v, \tag{3.1c}$$

for some $c > 0$. Then

(a) the probability measure $\bar{\nu}_c^\alpha$ *defined by*

$$\bar{\nu}_c^\alpha\{\zeta \in \{0,1\}^{\mathbb{N}} \colon \zeta_1 = \eta_1, \ldots, \zeta_n = \eta_n\} = \frac{w^T\left(\prod_{k=1}^n \eta_k D + (1-\eta_k)E\right)v}{w^T(D+E)^n v} \tag{3.2}$$

is invariant for the generator (2.2) *if and only if*

- *either* $\alpha \le \frac{1}{2}$ *and* $0 \le c \le \alpha(1-\alpha)$
- *or* $\alpha > \frac{1}{2}$ *and* $0 \le c \le \dfrac{1}{4}$.

(b) The measure $\bar{\nu}_c^\alpha$ *has stationary current* $\mathbb{E}_{\bar{\nu}_c^\alpha}[\eta_k(1-\eta_{k+1})] = c$*, for all* $k \ge 1$*. It equals* ν_α *if* $c = \alpha(1-\alpha)$ *and* $\alpha \le \frac{1}{2}$*, and otherwise it is asymptotically product with density* ϱ *given as the solution of* $c = \varrho(1-\varrho)$ *which satisfies* $\varrho \ge \dfrac{1}{2}$.

We now have two theorems showing invariant measures for the semi-infinite TASEP, namely Theorem 2.2 and Theorem 3.1. Both theorems depend on the injection rate α, however, the former depends on the initial asymptotic density ρ and the latter on the claimed stationary current c. The summary of both results can be seen on Table 3.1 and their parameter spaces in figure 3-2. The natural question is whether the stationary measures proposed by both theorems are the same. Note that the answer to this question does not follow directly from applying Theorem 2.2 to the measures of Theorem 3.1 when $\alpha > \frac{1}{2}$ since they are not product and hence do not satisfy the former's hypotheses.

Region	Injection rate α	Initial density ρ	Asymptotic density ϱ	Stationary current c	Liggett's stationary measure	Großkinsky's stationary measure
1	$0 \le \alpha \le \frac{1}{2}$	$0 \le \rho \le 1-\alpha$	α	$\alpha(1-\alpha)$	ν_α	ν_α
2	$0 \le \alpha \le \frac{1}{2}$	$1-\alpha < \rho \le \frac{1}{2}$	ρ	$\rho(1-\rho)$	μ_ρ^α	$\bar{\nu}_{\rho(1-\rho)}^\alpha$
3	$\frac{1}{2} < \alpha \le 1$	$\alpha \le \rho \le 1$	ρ	$\rho(1-\rho)$	μ_ρ^α	$\bar{\nu}_{\rho(1-\rho)}^\alpha$
4	$\frac{1}{2} < \alpha \le 1$	$\frac{1}{2} < \rho < \alpha$	ρ	$\rho(1-\rho)$	μ_ρ^α	$\bar{\nu}_{\rho(1-\rho)}^\alpha$
5	$\frac{1}{2} < \alpha \le 1$	$0 \le \rho \le \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\mu_{1/2}^\alpha$	$\bar{\nu}_{1/4}^\alpha$

Table 3.1: Summary of Theorems 2.2 and 3.1

(a) Injection rate and initial density (b) Injection rate and stationary current

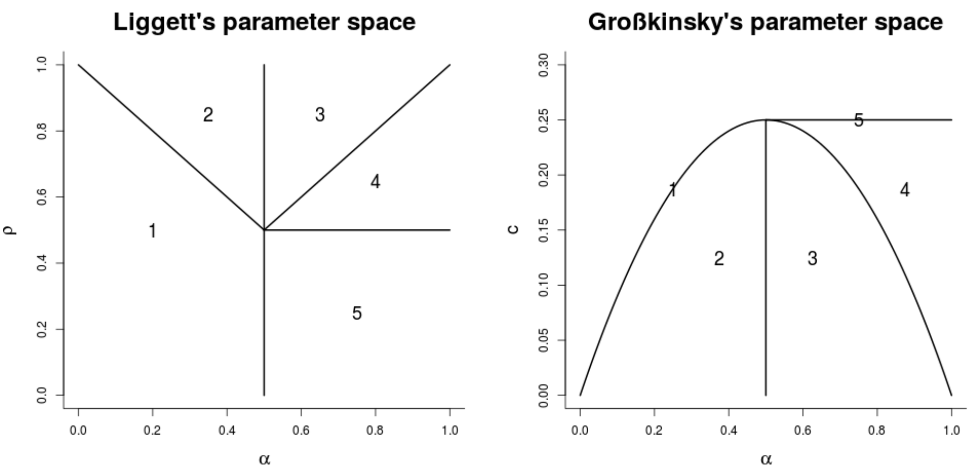

Figure 3-2: Relation of the parameter spaces of Theorems 2.2 and 3.1.

However, both theorems coincide at least on region 1 when the stationary distributions are product measures. Our next result concludes that Liggett's and Großkinsky's stationary measures coincide for regions 3, 4, and 5 as well. So far we are missing an analogue of Proposition 3.2 below for region 2 and therefore it will be out of the reach of this work bringing an opportunity to understand this case better in further research. We can see that this region is special since we may already observe a phase transition or order zero, see figure 3-3 where we have plotted the asymptotic density and stationary current as a function of the initial density given a fixed value of α.

Proposition 3.2. *If* $\alpha \geq \dfrac{1}{2}$, $\varrho \geq \dfrac{1}{2}$ *and* $c = \varrho(1-\varrho)$, *then the measures* $\bar{\nu}_c^\alpha$ *and* μ_ϱ^α *agree.*

Proof. By part (e) in [22, Theorem 3.10] the measure μ_ϱ^α is uniquely determined by the following two properties, numbered as in [22],

(c) If $u, n \in \mathbb{N}$ with $1 < u < u+1 < n$, and $\eta \in \{0,1\}^n$ with $\eta_u = 1$, $\eta_{u+1} = 0$, then

$$\mu_\varrho^\alpha\{\zeta : \zeta_k = \eta_k \text{ for } k \leq n\}$$
$$= c\,\mu_\varrho^\alpha\{\zeta : \zeta_k = \eta_k \text{ for } k \leq u-1, \zeta_k = \eta_{k+1} \text{ for } u+1 \leq k \leq n-1\}.$$

(d) If $n > 1$ and $\eta \in \{0,1\}^n$ with $\eta_1 = 0$, then

$$\alpha \mu_\varrho^\alpha\{\zeta : \zeta_k = \eta_k \text{ for } k \le n\} = c\, \mu_\varrho^\alpha\{\zeta : \zeta_k = \eta_{k+1} \text{ for } k \le n-1\}.$$

We show that $\bar{\nu}_c^\alpha$ satisfies these properties. Under the assumptions of (c) we get from properties (3.1a) in the second equality and (3.1c) in the third one

$$\begin{aligned}
&\bar{\nu}_c^\alpha\{\zeta : \zeta_k = \eta_k \text{ for } k \le n\} \\
&\qquad = \frac{w^T \big(\prod_{k=1}^{u-1} \eta_k D + (1-\eta_k)E \big) DE \big(\prod_{k=u+2}^{n} \eta_k D + (1-\eta_k)E \big) v}{w^T (D+E)^n v} \\
&\qquad = \frac{w^T \big(\prod_{k=1}^{u-1} \eta_k D + (1-\eta_k)E \big) (D+E) \big(\prod_{k=u+2}^{n} \eta_k D + (1-\eta_k)E \big) v}{w^T (D+E)^n v} \\
&\qquad = c\, \bar{\nu}_c^\alpha\{\zeta : \zeta_k = \eta_k \text{ for } k \le u-1, \zeta_k = \eta_{k+1} \text{ for } u+1 \le k \le n-1\}.
\end{aligned}$$

Under the assumptions of (d) we get from conditions (3.1b) in the second equality and (3.1c) in the third one,

$$\begin{aligned}
\alpha \bar{\nu}_c^\alpha\{\zeta : \zeta_k = \eta_k \text{ for } k \le n\} &= \alpha \frac{w^T E \left(\prod_{k=2}^{n} \eta_k D + (1-\eta_k)E \right) v}{w^T (D+E)^n v} \\
&= \frac{w^T \left(\prod_{k=2}^{n} \eta_k D + (1-\eta_k)E \right) v}{w^T (D+E)^n v} \\
&= c\, \bar{\nu}_c^\alpha\{\zeta : \zeta_k = \eta_{k+1} \text{ for } k \le n-1\}.
\end{aligned}$$

Hence $\bar{\nu}_c^\alpha$ satisfies (c) and (d) and therefore agrees with μ_ϱ^α. ■

Summarising what we have learnt so far, Theorem 2.2 shows the existence of stationary distributions provided we start from product measures. Theorem 3.1 says there are also stationary distributions that may possibly be written as matrix products. Proposition 3.2 proves that, for all five regions except number 2, the measures of both theorems agree. We should also point out the fact that so far we have no reason to split regions 3 and 4, this distinction will become clear until the next section.

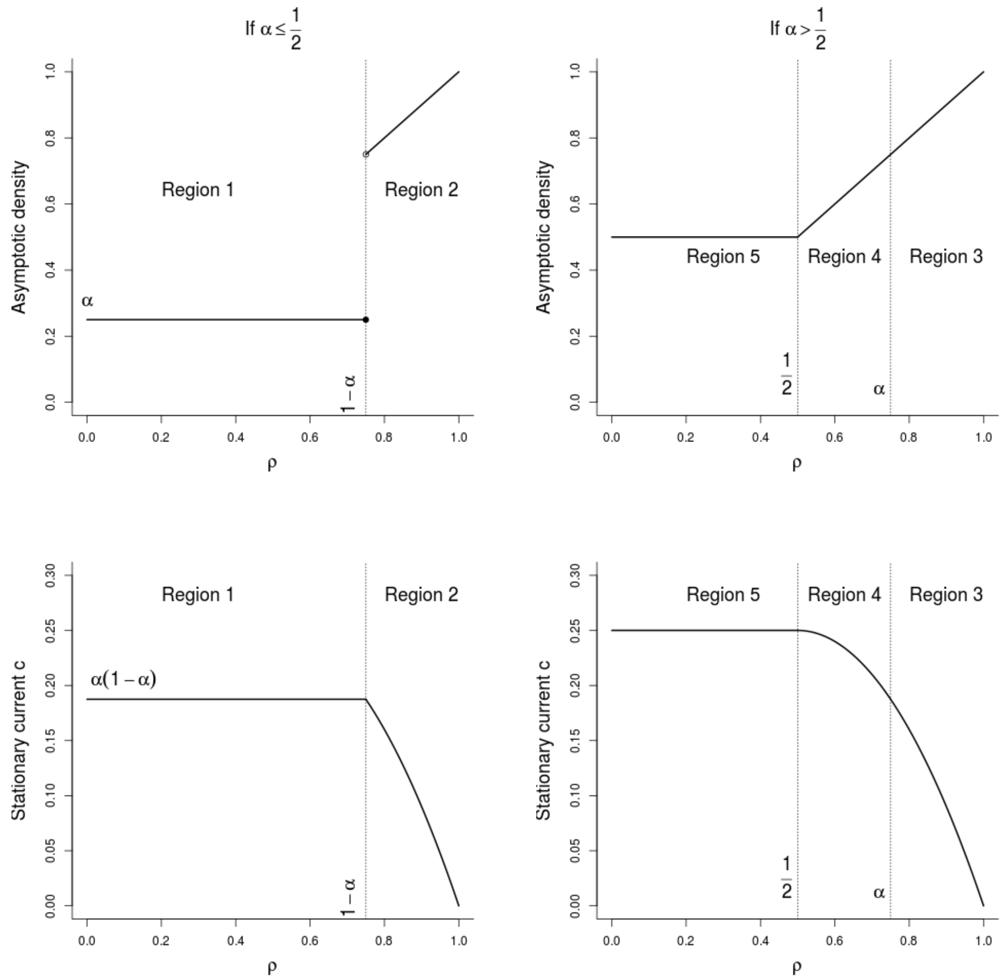

Figure 3-3: **Top left:** For the case $\alpha \leq \frac{1}{2}$, there is a clear phase transition of order zero as soon as the initial density goes above $1 - \alpha$. This is region 2 on the parameter space and we will leave this region out of further analysis. **Bottom left:** In contrast with the asymptotic density, the stationary current changes continuously, but we still see the phase transition from region 1 to region 2. **Top right:** For the case $\alpha > \frac{1}{2}$, we can also see a phase transition of order one between regions 4 and 5 as the curve shown is not differentiable. **Bottom right:** We can observe that the change in the stationary current between regions 4 and 5 is smoother here.

3.2 A solution to the semi-infinite TASEP

We now address the question of whether an explicit form of the matrices and vectors Theorem 3.1 talks about actually exist.

Proposition 3.3. *Let D and E be matrices and w and v vectors satisfying conditions (3.1). If $DE = ED$ then the matrices D and E and vectors w and v may be chosen all as scalars.*

Proof. Assume $DE = ED$ and applying conditions (3.1) we have

$$\begin{aligned}
\frac{w^T v}{\alpha c} &= w^T E(D+E)v \\
&= w^T (ED + E^2)v \\
&= w^T (DE + E^2)v \\
&= w^T (D+E)v + w^T E^2 v \\
&= \left(\frac{1}{c} + \frac{1}{\alpha^2}\right) w^T v.
\end{aligned}$$

Note that $w^T v \neq 0$ since otherwise measure (3.2) could not be defined. Hence, dividing by $w^T v$ we have $c = \alpha(1-\alpha)$. By Theorem 3.1 this only occurs on region 1 of figure 3-2 which is a product measure with constant density α.

It is a straightforward calculation to show that choosing $w = v = 1$, $D = \dfrac{1}{1-\alpha}$ and $E = \dfrac{1}{\alpha}$ indeed yields the desired product measure.

∎

Proposition 3.4. *Let D and E be matrices and w and v vectors satisfying conditions (3.1). If $DE \neq ED$ then they must be infinite dimensional matrices and hence w and v are also infinite dimensional vectors.*

Proof. Assume for contradiction that D and E are finite dimensional, say of dimension n. Suppose there exists a vector $x \neq 0$ such that $Ex = x$ then

$$(D+E)x = (D+E)Ex = (D+E)x + E^2 x = (D+E)x + x$$

This implies that $x = 0$, hence $(E - I)x = 0$ if and only if $x = 0$. Since $E - I$ is a finite dimensional matrix then it is invertible and

$$I = (E - I)(E - I)^{-1} = E(E - I)^{-1} - (E - I)^{-1}. \tag{3.3}$$

Using Equation 3.3 and condition (3.1a) we have

$$DE = D + E = E(E - I)^{-1} + E = E[I + (E - I)^{-1}] = EE(E - I)^{-1} = ED,$$

contradicting our original assumption. Therefore D and E are infinite dimensional. ■

To show a explicit solution of Theorem 3.1, we first define the values

$$\lambda_1 = \frac{1 - 2c + \sqrt{1 - 4c}}{2c}, \tag{3.4}$$

$$\lambda_2 = \frac{1 - 2c - \sqrt{1 - 4c}}{2c}. \tag{3.5}$$

Elementary calculations show that the matrices D, E and the vectors v and w defined by

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.6a}$$

$$w^T = \left(1, \tfrac{1}{\alpha} - 1, \left(\tfrac{1}{\alpha} - 1\right)^2, \cdots\right) \text{ and } v = \begin{pmatrix} 1 \\ \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1 - \lambda_2} \\ \frac{\lambda_1^3 - \lambda_2^3}{\lambda_1 - \lambda_2} \\ \vdots \end{pmatrix} \tag{3.6b}$$

satisfy the matrix product conditions (3.1) provided that $w^T v$ converges as a series.

Proposition 3.5. *Let w and v be defined as in (3.6). If parameters α and c are taken* *from regions* 4 *or* 5*, as in figure 3-2b then the series* $w^T v$ *converges.*

Proof. This proof is divided in 3 parts: The first part will find bounds for λ_1. The second part will find bounds for λ_2. Recall these values were defined by (3.4). The third part will use these bounds to show the convergence required.

Take α and c in regions 4 or 5 from figure 3-2b. Then $\frac{1}{2} < \alpha < 1$ and

$$\alpha(1-\alpha) < c \leq \frac{1}{4}. \tag{3.7}$$

Multiplying (3.7) by 2, subtracting 1 and changing signs we get

$$\frac{1}{2} \leq 1 - 2c < 1 - 2\alpha(1-\alpha). \tag{3.8}$$

Similarly, multiplying (3.7) by 4, subtracting 1 and changing signs we get

$$0 \leq 1 - 4c < (2\alpha - 1)^2. \tag{3.9}$$

Since $2\alpha > 1$ we can take the square root of (3.9) to obtain

$$0 \leq \sqrt{1-4c} < 2\alpha - 1. \tag{3.10}$$

Now multiply (3.7) by 2 and consider the multiplicative inverses to get

$$2 \leq \frac{1}{2c} < \frac{1}{2\alpha(1-\alpha)}. \tag{3.11}$$

Multiply the sum of (3.8) and (3.10) by (3.11) to find

$$1 \leq \frac{1 - 2c + \sqrt{1-4c}}{2c} < \frac{\alpha}{1-\alpha}. \tag{3.12}$$

Multiplying (3.12) by $\frac{1}{\alpha} - 1$ yields the end of the first part of the proof

$$\frac{1}{\alpha} - 1 \leq \left(\frac{1}{\alpha} - 1\right) \lambda_1 < 1. \tag{3.13}$$

To find the bounds on λ_2, note from (3.7) that $1 - 4c \geq 0$. Adding $4c^2$

$$1 - 4c + 4c^2 > 1 - 4c \geq 0, \tag{3.14}$$

completes a perfect square. Taking the square root and subtracting we find

$$1 - 2c - \sqrt{1 - 4c} > 0. \tag{3.15}$$

Once again multiply (3.11) now by (3.15) to get

$$\lambda_2 = \frac{1 - 2c - \sqrt{1 - 4c}}{2c} > 0.$$

Since $1 - 2c - \sqrt{1 - 4c} \leq 1 - 2c + \sqrt{1 - 4c}$, multiplying by $\frac{1}{\alpha} - 1$ and using (3.13) we end our second part with

$$0 < \left(\frac{1}{\alpha} - 1\right) \lambda_2 \leq \left(\frac{1}{\alpha} - 1\right) \lambda_1 < 1. \tag{3.16}$$

Finally, consider the series

$$\begin{aligned}
w^T v &= \sum_{k \in \mathbb{N}} w_k v_k & (3.17)\\
&= \sum_{k \in \mathbb{N}} \left(\frac{1}{\alpha} - 1\right)^{k-1} \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} & (3.18)\\
&= \frac{\alpha}{(1-\alpha)(\lambda_1 - \lambda_2)} \left\{ \sum_{k \in \mathbb{N}} \left[\left(\frac{1}{\alpha} - 1\right) \lambda_1\right]^k - \sum_{k \in \mathbb{N}} \left[\left(\frac{1}{\alpha} - 1\right) \lambda_2\right]^k \right\} & (3.19)\\
&< \infty. & (3.20)
\end{aligned}$$

The convergence follows since by (3.16) the two sums are convergent geometric series. ∎

This is the reason that makes region 3 different from region 4. The matrices and vectors defined by (3.6) do not apply on region 3 while they work on regions 4 and 5. We will therefore omit region 3 from the rest of this work and realise that region 1 corresponds to part *(a)* of Theorem 2.3. Likewise, regions 5 and 4 correspond to parts *(b)* and *(c)* respectively.

Now that we know an explicit form of the stationary measure, we proceed to find the rate function of a large deviation principle.

3.3 An application of the Gärtner-Ellis Theorem

Fortunately, the theory of large deviations has been largely developed and we do not need to check that a function satisfies the conditions given by the definition of large deviations every time. We will take an alternative route via an application of Gärtner-Ellis Theorem, see Theorem V.6 in [11].

Theorem 3.6. *(Gärtner-Ellis) Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables on a probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$, where $\mathcal{X}$ is a nonempty subset of $\mathbb{R}$. If the limit cumulant generating function $\Lambda\colon \mathbb{R} \to \mathbb{R}$ defined by*

$$\Lambda(\theta) = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}[e^{n\theta X_n}]$$

exists and is differentiable on all $\mathbb{R}$, then $\{X_n\}_{n\in\mathbb{N}}$ satisfies a large deviation principle with rate function $I\colon \mathcal{X} \to [-\infty, \infty]$ defined by

$$I(x) = \sup_{\theta\in\mathbb{R}} \{x\theta - \Lambda(\theta)\}.$$

Depending on whether the parameters α and c are in regions 1, 4 or 5 the stationary measure will be ν_α, $\bar{\nu}_c^\alpha$ or $\bar{\nu}_{1/4}^\alpha$. Denote by $\bar{\nu}^{(r)}$ with $r \in \{1, 4, 5\}$ the corresponding stationary measure for region r. To calculate the moment generating function $M_n(\theta)$ of X_n we use Theorem 3.1 and Proposition 3.2 in the third equality and condition (3.1c) in the fifth one, to get

$$\begin{aligned}
M_n(\theta) &= \mathbb{E}\big[e^{n\theta X_n}\big] = \mathbb{E}\Big[\exp\big(\theta\sum_{k=1}^{n}\xi_k\big)\Big] \\
&= \sum_{\eta\in\{0,1\}^n} \bar{\nu}^{(r)}\{\xi : \xi_k = \eta_k \text{ for } k \le n\} \exp\left(\theta\sum_{k=1}^{n}\eta_k\right) \\
&= \frac{w^T(e^\theta D + E)^n v}{w^T(D+E)^n v} = \frac{c^n}{w^T v} w^T(e^\theta D + E)^n v,
\end{aligned}$$

and the limit cumulant generating function simplifies to

$$\begin{aligned}
\Lambda(\theta) &= \lim_{n\to\infty} \tfrac{1}{n} \log M_n(\theta) \\
&= \lim_{n\to\infty} \tfrac{1}{n} \log w^T(e^\theta D + E)^n v + \log c.
\end{aligned} \tag{3.21}$$

If D and E were finite matrices, we could identify this limit using the Perron-Frobenius theorem as the spectral radius of the matrix $e^\theta D + E$. However in our example (and in almost all physically interesting examples) the matrices solving (3.1) are necessarily infinite.

3.4 Product measures

The only scenario under which matrices D and E are finite happens when choosing parameters from region 1, that is $\alpha \le \frac{1}{2}$ and $c = \alpha(1-\alpha)$. In the previous section we saw that we may choose $w = v = 1$, $D = \frac{1}{1-\alpha}$, $E = \frac{1}{\alpha}$. Therefore the limit cumulant generating function simplifies to

$$\Lambda(\theta) = \log(\alpha e^\theta + 1 - \alpha) \tag{3.22}$$

Taking the Legendre transform we find

$$
\begin{aligned}
I(x) &= \sup_{\theta\in\mathbb{R}}\{x\theta - \Lambda(\theta)\} \\
&= \sup_{\theta\in\mathbb{R}}\{x\theta - \log(\alpha e^{\theta} + 1 - \alpha)\} \\
&= x\log\frac{x}{\alpha} + (1-x)\log\frac{1-x}{1-\alpha}.
\end{aligned} \tag{3.23}
$$

This is the rate function of the large deviation principle of the average of independent and identically distributed Bernoulli random variables with parameter α and therefore we have proven part (a) of Theorem 2.3. The rest of the thesis will focus on proving parts (b) and (c).

3.5 Asymptotic product measures

If parameters α and c are chosen from regions 4 or 5 the matrices D and E are necessarily infinite and consequently so are w and v. A first idea would be to truncate the matrices to finite size, calculate the spectral radius and take a limit, but this turns out to lead to a wrong result, as it neglects the important information contained in the vectors v and w.

3.5.1 A failed attempt: Finite matrices approximation

Consider the infinite matrices D and E defined by (3.6) and for $\theta \in \mathbb{R}$ define the infinite matrix $A(\theta) = e^{\theta}D + E$. For $m \in \mathbb{N}$ define the finite matrix $A_m(\theta)$ as the first $2^m \times 2^m$ block of $A(\theta)$. That is, the $2^m \times 2^m$ matrix with components

$$
A_m(\theta)_{k,j} = \begin{cases} e^{\theta} & \text{if } j = k+1, \\ 1 + e^{\theta} & \text{if } j = k, \\ 1 & \text{if } j = k-1, \\ 0 & \text{else.} \end{cases}
$$

Denote by $\mathcal{E}_m(\theta)$ the set of eigenvalues of $A_m(\theta)$ and by $\zeta_m(\theta)$ the largest eigenvalue of $A_m(\theta)$.

Conjecture 3.7. *For $m \in \mathbb{N}$, the cardinality of $\mathcal{E}_m(\theta)$ is 2^m and the largest eigenvalue of $A_m(\theta)$ is given by*

$$\zeta_m(\theta) = e^\theta + 1 + \sqrt{2e^\theta + \sqrt{2e^{2\theta} + \sqrt{2e^{4\theta} + \ldots + \sqrt{2e^{2^{m-2}\theta}}}}}.$$

Provided the conjecture is true, then there exist matrices $\{U_\lambda\}_{\lambda \in \mathcal{E}_m(\theta)}$ such that for all $n \in \mathbb{N}_0$

$$A_m(\theta)^n = \sum_{\lambda \in \mathcal{E}_m(\theta)} \lambda^n U_\lambda.$$

It is a straightforward calculation to show that for any pair vectors $a, b \in \mathbb{R}^{2^m}$

$$\lim_{n\to\infty} \tfrac{1}{n} \log a^T A_m(\theta)^n b = \log \zeta_m(\theta)$$

Following equation (3.21), the idea goes as follows: In the second equality, approximate the infinite matrix $A(\theta)$ by the finite one $A_m(\theta)$. In the third equality, we need to find a sequence of vectors that approximate w and v and by continuity of the logarithm bring the limit outside of it, however this needs a solid argument. The fourth line also needs a proper justification to interchange the limits. The fifth equality simplifies the problem to find the limit of the sequence of eigenvalues.

$$\begin{aligned}
\Lambda(\theta) &= \lim_{n\to\infty} \frac{1}{n} \log w^T A(\theta)^n v + \log c \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left(w^T \lim_{m\to\infty} A_m(\theta)^n v \right) + \log c \\
&\approx \lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{n} \log w^{(m)T} A_m(\theta)^n v^{(m)} + \log c \\
&\approx \lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{n} \log w^{(m)T} A_m(\theta)^n v^{(m)} + \log c \\
&= \lim_{m\to\infty} \zeta_m(\theta) + \log c.
\end{aligned}$$

It can be proved that the limit of the eigenvalues is

$$\lim_{m\to\infty} \zeta_m(\theta) = (1 + e^{\frac{\theta}{2}})^2.$$

This result would imply that

$$\Lambda(\theta) = 2\log(1 + e^{\frac{\theta}{2}}) + \log c,$$

independently of α and hence giving a rate function

$$I(x) = 2[x \log x + (1 - x)\log(1 - x) - log\sqrt{c}].$$

However, this can only be a rate function if $c = \frac{1}{4}$.

In summary, this approach has three main caveats: First, although there is numerical evidence supporting Conjecture 3.7 we still need a proof. Second, the exchange of limits, power and size of the matrix, needs to be justified, including the vector approximations to w and v. Finally, we know the zero of the rate function by Theorem 2.2 and therefore the functional found here is wrong.

3.5.2 Divide and conquer: Upper and lower bounds

Summarising what we have learnt in this chapter: We saw that the stationary measures given by Theorem 2.2 can be written as matrix products for suitable initial product measures and injection rates. We explicitly showed these matrices in (3.6). In finding a large deviation principle, our attempt to simplify explicitly Equation (3.21) was not successful.

In the next two chapters we will find lower and upper bounds for $\Lambda(\theta)$. The approaches taken for each of these bounds are very different, nevertheless, and somewhat surprisingly, we will find that the bounds coincide. This is where the beauty, novelty and generality of our method can be appreciated. The upper bound comes from a very basic knowledge of weighted ℓ^2 spaces used in functional analysis and we exploit the Toeplitz form of our operator $A(\theta)$. The lower bound is found via combinatorial calculations.

CHAPTER 4

UPPER BOUND: SPECTRAL THEORY OF TOEPLITZ OPERATORS

Recall that our main objective is to find the rate function of a large deviation principle for the sequence (2.3). To do this we seek to use Theorem 3.6 and hence we first need an expression for Equation (3.21). In this chapter we will find an upper bound for $\Lambda(\theta)$.

We will exploit that matrices D and E, as well as the vectors v and w, solving (3.1) are explicitly known. We introduce weighted ℓ^2 spaces, denoted ℓ^2_s, and interpret the matrix $A(\theta) = e^\theta D + E$ as an operator on these spaces. If the weights are such that v is an element of ℓ^2_s, and w an element of its dual, we can get a bound on (3.21) from the spectral radius of the operator, which can be optimised by minimising the bound over all admissible weights. In order to obtain the spectral radius we use a simple isomorphism between weighted and unweighted ℓ^2 spaces. Acting on the unweighted spaces, the operator has a Toeplitz structure and from the general theory of Toeplitz operators on ℓ^2 an explicit formula for the spectral radius is available.

4.1 Weighted ℓ^2 spaces

Let $s > 0$ and consider the weighted spaces

$$\ell^2_s = \{x = (x_k)_{k\in\mathbb{N}} : \sum_{k\in\mathbb{N}} |x_k|^2 s^k < \infty.\}$$

Note that choosing $s = 1$ recovers the usual ℓ^2 space. Moreover for fixed s, ℓ^2_s with its corresponding norm

$$|x|^2_{\ell^2_s} = \sum_{k\in\mathbb{N}} |x_k|^2 s^k$$

is a Banach space. The next lemma will help us to translate classic ℓ^2 theory to ℓ^2_s.

Lemma 4.1. *The function $T_s \colon \ell^2 \to \ell^2_s$ defined by*

$$(T_s x)_k = \frac{x_k}{s^{k/2}}$$

for $s > 0$ is a bijective isometry.

Proof. We can define the inverse $T_s^{-1} \colon \ell^2_s \to \ell^2$ by $(T_s^{-1}x)_k = x_k s^{k/2}$ and hence T_s is bijective. We just need to prove it is an isometry, so let $x \in \ell^2_s$ and calculate

$$|T_s^{-1}x|^2_{\ell^2} = \sum_{k\in\mathbb{N}} |(T_s^{-1}x)_k|^2 = \sum_{k\in\mathbb{N}} |x_k s^{k/2}|^2 = |x|^2_{\ell^2_s}.$$

Analogously, for $x \in \ell^2$ we have $|T_s x|_{\ell^2_s} = |x|_{\ell^2}$. ∎

We now look into the structure of the dual space of ℓ^2_s space.

Lemma 4.2. *The dual space ℓ^{2*}_s can be identified with $\ell^2_{s^{-1}}$.*

Proof. Define the dual product $\langle \cdot, \cdot \rangle_D \colon \ell^2_{s^{-1}} \times \ell^2_s \to \mathbb{R}$ by $\langle y, x \rangle_D = \langle T^{-1}_{s^{-1}} y, T_s^{-1} x \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in ℓ^2. We first prove that for each vector $y \in \ell^2_{s^{-1}}$ there exists a function $f_y \in \ell_s^{2*}$ such that $f_y(x) = \langle y, x\rangle_D$. To this end, let $y \in \ell_{s^{-1}}^2$ and define $f_y \colon \ell_s^2 \to \mathbb{R}$ by

$$f_y(x) = \langle y, x\rangle_D = \sum_{k\in\mathbb{N}} x_k y_k.$$

The linearity of f_y follows easily from the definition; the Cauchy-Schwarz inequality in ℓ^2 shows it is also bounded,

$$\begin{aligned} |f_y(x)| &= |\sum_{k\in\mathbb{N}} x_k y_k| = |\sum_{k\in\mathbb{N}} x_k s^{\frac{k}{2}} y_k s^{-\frac{k}{2}}| \\ &\le \left(\sum_{k\in\mathbb{N}} |x_k|^2 s^k\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{N}} |y_k|^2 s^{-k}\right)^{\frac{1}{2}} = |x|_{\ell_s^2} |y|_{\ell_{s^{-1}}^2}. \end{aligned}$$

Conversely, let $f \in \ell_s^{2*}$. Define $g \colon \ell^2 \to \mathbb{R}$ by $g(x) = (f \circ T_s)(x)$. Since f and T_s are both linear, so is g, and since f is bounded,

$$|g(x)| = |(f \circ T_s)(x)| \le |f|_{\ell_s^{2*}} |T_s(x)|_{\ell_s^2} = |f|_{\ell_s^{2*}} |x|_{\ell^2} < \infty.$$

Hence, $g \in \ell^{2*}$ and by the Riesz Representation Theorem there exists a unique $\tilde{y} \in \ell^2$ such that $g(x) = \langle x, \tilde{y}\rangle$ for all $x \in \ell^2$. Let $y = T_{s^{-1}}\tilde{y} \in \ell_{s^{-1}}^2$. Since T_s is invertible we have that for all $x \in \ell_s^2$

$$f(x) = (g \circ T_s^{-1})(x) = \langle T_s^{-1}x, \tilde{y}\rangle = \sum_{k\in\mathbb{N}} (T_s^{-1}x)_k (T_{s^{-1}}^{-1} y)_k = \sum_{k\in\mathbb{N}} x_k y_k = \langle y, x\rangle_D,$$

therefore $f \in \ell_s^{2*}$ is represented by $y \in \ell_{s^{-1}}^2$. ■

Define for $\theta \in \mathbb{R}$ the operator $A(\theta) \colon \ell_s^2 \to \ell_s^2$ with the infinite matrix representation $e^\theta D + E$, where matrices D and E are defined by (3.6). Hence, the k-th component of the vector $A(\theta)x$ is

$$(A(\theta)x)_k = \begin{cases} x_1(1 + e^\theta) + x_2 e^\theta & \text{if } k = 1 \\ x_{k-1} + x_k(1 + e^\theta) + x_{k+1}e^\theta & \text{if } k > 1. \end{cases}$$

Proposition 4.3. *The operator $A(\theta)\colon \ell_s^2 \to \ell_s^2$ is bounded.*

Proof. Let $x \in \ell_s^2$. Using Cauchy-Schwarz in $\mathbb{R}^2$ and $\mathbb{R}^3$ for each term of $A(\theta)x$ gives

$$\begin{aligned}
|A(\theta)x|_{\ell_s^2}^2 &= \sum_{k\in\mathbb{N}} |(A(\theta)x)_k|^2 s^k \\
&= |x_1(1+e^\theta) + x_2 e^\theta|^2 s + \sum_{k=2}^{\infty} |x_{k-1} + x_k(1+e^\theta) + x_{k+1}e^\theta|^2 s^k \\
&\le (x_1^2 + x_2^2)((1+e^\theta)^2 + e^{2\theta}) + \sum_{k=2}^{\infty} (x_{k-1}^2 + x_k^2 + x_{k+1}^2)(1 + (1+e^\theta)^2 + e^{2\theta}) s^k \\
&\le C_s(\theta)|x|_{\ell_s^2}^2,
\end{aligned}$$

where $C_s(\theta) > 0$ is a constant independent of x and hence we see that $A(\theta)$ is a bounded linear operator. ■

Lemma 4.4. *Let $L \in \mathcal{L}(\ell_s^2)$, that is a bounded linear operator from ℓ_s^2 to itself. The operator $\tilde{L} = T_s^{-1} \circ L \circ T_s$ satisfies $\tilde{L} \in \mathcal{L}(\ell^2)$.*

Proof. Take $x \in \ell^2$. Then by Lemma 4.1,

$$|\tilde{L}x|_{\ell^2} \le |T_s^{-1}|_{\mathcal{L}(\ell_s^2,\ell^2)} |L|_{\mathcal{L}(\ell_s^2)} |T_s|_{\mathcal{L}(\ell^2,\ell_s^2)} |x|_{\ell^2} < \infty.$$

By Lemma 4.1 we conclude that $|\tilde{L}|_{\mathcal{L}(\ell^2)} \le |L|_{\mathcal{L}(\ell_s^2)}$. Analogously, since $L = T_s \circ L \circ T_s^{-1}$, we have that $|\tilde{L}|_{\mathcal{L}(\ell^2)} = |L|_{\mathcal{L}(\ell_s^2)}$. ■

The tilde operator commutes with exponentiation.

Lemma 4.5. *Let $L \in \mathcal{L}(\ell_s^2)$, then $\widetilde{L^n} = \tilde{L}^n$.*

Proof. We proceed by induction over n. For $n = 1$, the proposition is a tautology. We assume the proposition true for n, let $x \in \ell^2$ and calculate

$$\tilde{L}^{n+1}x = \tilde{L} \circ \tilde{L}^n x = T_s^{-1} \circ L \circ T_s \circ T_s^{-1} \circ L^n \circ T_s x = T_s^{-1} \circ L^{n+1} \circ T_s x = \widetilde{L^{n+1}} x.$$

■

Recall the definitions of λ_1 and λ_2 in (3.4) and that in Proposition 3.5 we showed that $0 < \lambda_2 \leq \lambda_1$. Recall the explicit form of v in (3.6) and calculate its norm as an element of ℓ_s^2

$$|v|_{\ell_s^2}^2 = \sum_{k\in\mathbb{N}} |v_k|^2 s^k = \sum_{k\in\mathbb{N}} \left(\frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}\right)^2 s^k \tag{4.1}$$

$$= \frac{1}{(\lambda_1 - \lambda_2)^2}\left(\sum_{k\in\mathbb{N}} (s\lambda_1^2)^k - 2\sum_{k\in\mathbb{N}} s^k + \sum_{k\in\mathbb{N}} (s\lambda_2^2)^k\right), \tag{4.2}$$

which is finite if and only if $s < 1$, $s\lambda_1^2 < 1$, and $s\lambda_2^2 < 1$. These three conditions are satisfied by taking $s \in (0, \frac{1}{\lambda_1^2})$.

On the other hand, if $s > (\frac{1}{\alpha} - 1)^2$,

$$|w|_{\ell_{s^{-1}}^2}^2 = \sum_{k\in\mathbb{N}} \left(\frac{1}{\alpha} - 1\right)^{2(k-1)} s^{-k} = \frac{1}{s - (\frac{1}{\alpha} - 1)^2} < \infty. \tag{4.3}$$

Imposing the conditions of $v \in \ell_s^2$ and $w \in \ell_{s-1}^2$ restricts the set of weighted spaces we may consider to $s \in ((\frac{1}{\alpha} - 1)^2, \frac{1}{\lambda_1^2})$. Note that this set is non-empty for the regions of Figure 3-2 we are currently considering, that is regions 4 and 5.

4.2 Toeplitz operators

We now need to review some properties of Toeplitz operators before stating the main result of this section. Let $a = \{a_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C})$, that is, a double sequence of complex numbers such that $\sum_{k\in\mathbb{Z}} |a_k|^2 < \infty$. A Toeplitz operator A defined by the double sequence $a \in \ell^2(\mathbb{C})$ is an infinite matrix with the structure

$$A = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

The *symbol* $\kappa\colon \{z \in \mathbb{C} : |z| = 1\} \to \mathbb{C}$ of a Toeplitz operator is defined by

$$\kappa(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

We recall Theorem 7.1 in [28] that deals with spectra of Toeplitz operators.

Theorem 4.6. *[28, Theorem 7.1] Let A be a Toeplitz operator. If A has a continuous symbol κ, then its spectrum is given by the image of the unit circle under κ together with all the points enclosed by this curve with non-zero winding number.*

For fixed $\theta \in \mathbb{R}$, recall that the operator $A(\theta)$ has an infinite matrix representation given by

$$A(\theta) = e^{\theta} D + E = \begin{pmatrix} 1+e^{\theta} & e^{\theta} & 0 & 0 & \cdots \\ 1 & 1+e^{\theta} & e^{\theta} & 0 & \cdots \\ 0 & 1 & 1+e^{\theta} & e^{\theta} & \cdots \\ 0 & 0 & 1 & 1+e^{\theta} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{4.4}$$

which by Proposition 4.3 is in $\mathcal{L}(\ell^2_s)$. By Lemma 4.4 the operator $\tilde{A}(\theta)$ is a Toeplitz operator in ℓ^2 with its symbol κ given by

$$\kappa(z) = \frac{e^{\theta}}{z\sqrt{s}} + 1 + e^{\theta} + z\sqrt{s}.$$

Writing $z = e^{i\varphi}$ as an element of the unit circle, its symbol

$$\kappa(e^{i\varphi}) = 1 + e^{\theta} + \left(\sqrt{s} + \frac{e^{\theta}}{\sqrt{s}}\right)\cos\varphi + \left(\sqrt{s} - \frac{e^{\theta}}{\sqrt{s}}\right) i \sin\varphi,$$

is recognised as a parametrised ellipse centred at $1 + e^{\theta}$, with major axis of length $\sqrt{s} + \dfrac{e^{\theta}}{\sqrt{s}}$ along the real line, and minor axis of length $|\sqrt{s} - \dfrac{e^{\theta}}{\sqrt{s}}|$. Figure 4-1 shows the behaviour of these ellipses as the value of θ varies.

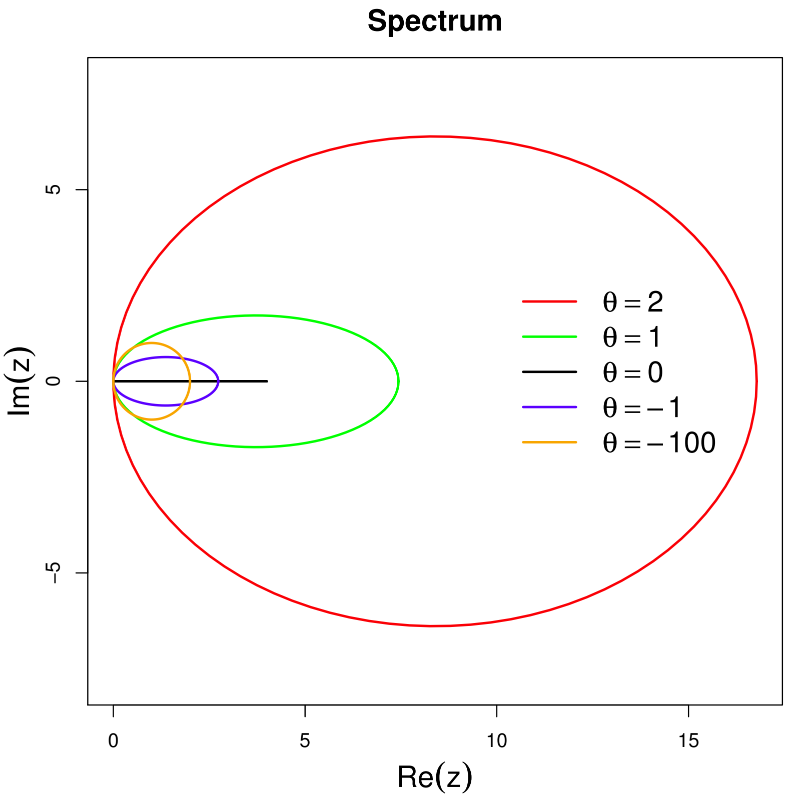

Figure 4-1: Boundary of the spectra of operators $\tilde{A}(\theta)$ for different values of θ when $s = 1$.

Therefore, the spectral radius of $\tilde{A}(\theta)$ is found evaluating the symbol at $z = 1$,

$$\rho(\tilde{A}(\theta)) = \kappa(1) = 1 + \sqrt{s} + e^{\theta}\left(1 + \frac{1}{\sqrt{s}}\right). \tag{4.5}$$

4.3 Optimising the upper bound

We now state the main result of this chapter: the upper bound for the limit cumulant generating function Λ.

Proposition 4.7. *For Λ defined by* (3.21)*, an upper bound is*

$$\Lambda(\theta) \leq \begin{cases} \log\left(\frac{e^\theta}{1-\alpha} + \frac{1}{\alpha}\right) + \log c & \text{if } -\infty < \theta \leq 2\log\left(\frac{1}{\alpha} - 1\right), \\ \log\left(1 + e^{\theta/2}\right)^2 + \log c & \text{if } 2\log\left(\frac{1}{\alpha} - 1\right) < \theta \leq -2\log\lambda_1, \\ \log\left(1 + \lambda_1 e^\theta\right) + \log\left(1 + \frac{1}{\lambda_1}\right) + \log c & \text{if } -2\log\lambda_1 < \theta < \infty. \end{cases}$$

Proof. Recall from (4.1) that $v \in \ell_s^2$; by Proposition 4.3 $A(\theta) \in \mathcal{L}(\ell_s^2)$ and therefore $A(\theta)v \in \ell_s^2$. Also $w \in \ell_{s-1}^2$ from (4.3). Hence, by (3.21) and Cauchy-Schwarz,

$$\begin{aligned} \Lambda(\theta) &= \lim_{n\to\infty} \tfrac{1}{n} \log(w^T A(\theta)^n v) + \log c \\ &\leq \lim_{n\to\infty} \tfrac{1}{n} \log(|w|_{\ell_{s-1}^2} |A(\theta)^n v|_{\ell_s^2}) + \log c. \end{aligned}$$

.

The norm of w does not contribute to the limit since it does not depend on n. By Lemma 4.4, Lemma 4.1, and Lemma 4.5 we can continue the previous estimate

$$\begin{aligned} &= \lim_{n\to\infty} \tfrac{1}{n} \log(|T_s \circ \widetilde{A(\theta)^n} \circ T_s^{-1} v|_{\ell_s^2}) + \log c \\ &\leq \lim_{n\to\infty} \tfrac{1}{n} \log(|\tilde{A}(\theta)^n|_{\mathcal{L}(\ell^2)} |T_s^{-1} v|_{\ell^2}) + \log c; \end{aligned}$$

Once again, the norm of $T_s^{-1}v$ does not contribute to the limit since it does not depend on n. We insert the factor $\frac{1}{n}$ to the logarithm by continuity and use the definition of spectral radius

$$\begin{aligned} &= \lim_{n\to\infty} \log(|\tilde{A}(\theta)^n|_{\mathcal{L}(\ell^2)}^{\frac{1}{n}}) + \log c \\ &= \log \rho(\tilde{A}(\theta)) + \log c. \end{aligned}$$

We now use (4.5) to find the spectral radius

$$= \log\left[1+\sqrt{s}+e^{\theta}\left(1+\frac{1}{\sqrt{s}}\right)\right]+\log c.$$

Since the left hand side does not depend on s, it is a lower bound on the right hand side for s, so we take the infimum over the interval $\left((\frac{1}{\alpha}-1)^2, \frac{1}{\lambda_1^2}\right)$.

$$\Lambda(\theta) \leq \inf_{s\in\left(\left(\frac{1}{\alpha}-1\right)^2, \frac{1}{\lambda_1^2}\right)} \log\left[1+\sqrt{s}+e^{\theta}\left(1+\frac{1}{\sqrt{s}}\right)\right]+\log c.$$

Given θ, the value of s that reaches the infimum of this function is given by

$$s^* = \begin{cases} \left(\dfrac{1}{\alpha}-1\right)^2 & \text{if } -\infty < \theta \leq \log\left(\dfrac{1}{\alpha}-1\right)^2, \\ e^{\theta} & \text{if } \log\left(\dfrac{1}{\alpha}-1\right)^2 < \theta \leq -2\log\lambda_1, \\ \dfrac{1}{\lambda_1^2} & \text{if } -2\log\lambda_1 < \theta < \infty. \end{cases}$$

Plugging s^* into the formula gives the result of the lemma. ■

Proposition 4.7 is the upper bound of Λ. In the next chapter we will see that this is also a lower bound. Note that this function is continuous and differentiable, see Figure 4-2, this is important since it is a requirement of Theorem 3.6.

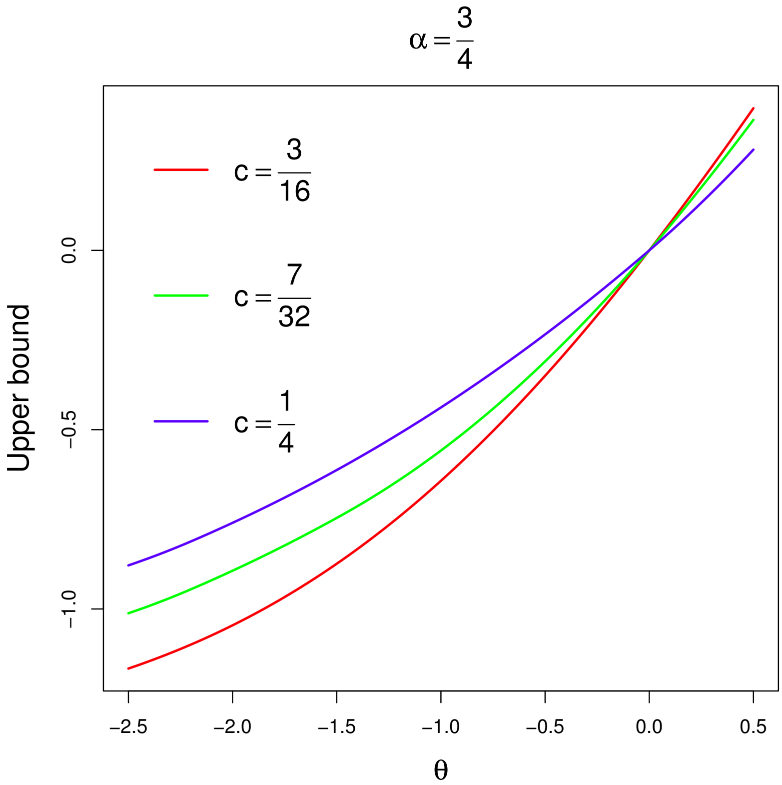

Figure 4-2: Plot of the upper bound of $\Lambda(\theta)$ for $\alpha = \frac{3}{4}$ and different values of c.

CHAPTER 5

LOWER BOUND: A COMBINATORIAL APPROACH

In our objective of finding the rate function of a large deviation principle for the sequence (2.3), we intend to apply Theorem 3.6 to the limit cumulant generating function (3.21). In the previous chapter we found an upper bound for Λ, in this chapter we claim that this upper bound is also a lower bound.

This chapter is divided in three sections. The first section deals with the expansion of $(e^{\theta}D + E)^n$ when D and E do not commute but they satisfy the conditions (3.1). The second section uses the combinatorial properties of the polynomials studied on the former section to establish a lower bound for the limit cumulant generating function. Finally, in the third section we formally apply Theorem 3.6 and find the rate function we are looking for.

5.1 Recursive polynomials

In order to find the lower bound we use a completely different approach. We will focus on the powers of $e^\theta D + E$. The first result we show is how to simplify $D^j E$.

Proposition 5.1. *Let D and E satisfy condition* (3.1a)*, then for $j \in \mathbb{N}$*

$$D^j E = \sum_{k=1}^{j} D^k + E.$$

Proof. We prove this by induction on j. For $j = 1$, $D^1 E = D + E$ by (3.1a). Assume the induction hypothesis $D^j E = \sum_{k=1}^{j} D^k + E$ and the next power, using (3.1a) again in the fourth equality, yields

$$\begin{aligned} D^{j+1}E &= DD^jE = D\big(\sum_{k=1}^{j} D^k + E\big) \\ &= \sum_{k=1}^{j} D^{k+1} + DE = \sum_{k=2}^{j+1} D^k + D + E \\ &= \sum_{k=1}^{j+1} D^k + E, \end{aligned}$$

as required by the induction step. ■

Proposition 5.2. *There exists a sequence of polynomials $f^n_{p,j}(\theta)$ on e^θ such that*

$$(e^\theta D + E)^n = \sum_{p=1}^{n} \sum_{j=0}^{p} f^n_{p,j}(\theta) E^{p-j} D^j \tag{5.1}$$

and they can be defined recursively in two ways: Starting with

$$f^1_{1,0}(\theta) = 1, \qquad f^1_{1,1}(\theta) = e^\theta,$$

the first characterisation for $f^n_{p,j}$ with $n > 1$ is

$$f^n_{p,j}(\theta) = \begin{cases} \displaystyle\sum_{k=1}^{n-1} f^{n-1}_{k,k}(\theta) & \text{if } n > 1, \quad p = 1, \quad j \in \{0, 1\} \\ \displaystyle f^{n-1}_{p-1,0}(\theta) + \sum_{k=p}^{n-1} f^{n-1}_{k,k-p+1}(\theta) & \text{if } n > 1, \quad 1 < p < n, \quad j = 0 \\ \displaystyle e^\theta f^{n-1}_{p-1,j-1}(\theta) + \sum_{k=p}^{n-1} f^{n-1}_{k,k-p+j}(\theta) & \text{if } n, p > 1, \quad j \le p < n, \quad j > 0 \\ f^{n-1}_{n-1,0}(\theta) & \text{if } n > 1, \quad p = n, \quad j = 0 \\ e^\theta f^{n-1}_{n-1,j-1}(\theta) & \text{if } n > 1, \quad p = n, \quad 0 < j \le p; \end{cases} \tag{5.2}$$

and the second characterisation is

$$f^n_{p,j}(\theta) = \begin{cases} \displaystyle\sum_{k=1}^{n-1} e^\theta f^{n-1}_{k,0}(\theta) & \text{if } n > 1, \quad p = 1, \quad j \in \{0, 1\} \\ \displaystyle f^{n-1}_{p-1,j}(\theta) + \sum_{k=p}^{n-1} e^\theta f^{n-1}_{k,j}(\theta) & \text{if } n > 1, \quad 1 < p < n, \quad 0 \le j < p \\ \displaystyle e^\theta f^{n-1}_{p-1,p-1}(\theta) + \sum_{k=p}^{n-1} e^\theta f^{n-1}_{k,p-1}(\theta) & \text{if } n > 1, \quad 1 < p < n, \quad j = p \\ f^{n-1}_{n-1,j}(\theta) & \text{if } n > 1, \quad p = n, \quad 0 \le j < n \\ e^\theta f^{n-1}_{n-1,n-1}(\theta) & \text{if } n > 1, \quad p = n, \quad j = n. \end{cases} \tag{5.3}$$

Proof. We prove this by induction. For $n = 1$ we have $(e^\theta D + E)^1 = e^\theta D + E$ which settles the initial values $f^1_{1,0}(\theta) = 1$ and $f^1_{1,1}(\theta) = e^\theta$. To find the recursion we assume the induction hypothesis:

$$(e^\theta D + E)^n = \sum_{p=1}^{n} \sum_{j=0}^{p} f^n_{p,j}(\theta) E^{p-j} D^j$$

and expand the next power. However, there are two ways we can use to expand, namely $(e^\theta D + E)^{n+1} = (e^\theta D + E)^n (e^\theta D + E)$ or $(e^\theta D + E)(e^\theta D + E)^n$.

We will start with the former

$$
\begin{aligned}
(e^\theta D + E)^{n+1} &= (e^\theta D + E)^n (e^\theta D + E) \\
&= \big(\sum_{p=1}^{n} \sum_{j=0}^{p} f_{p,j}^{n}(\theta) E^{p-j} D^j\big)(e^\theta D + E) \\
&= \sum_{p=1}^{n} \sum_{j=0}^{p} e^\theta f_{p,j}^{n}(\theta) E^{p-j} D^{j+1} + \sum_{p=1}^{n} \sum_{j=0}^{p} f_{p,j}^{n}(\theta) E^{p-j} D^j E,
\end{aligned}
$$

we now change the index $j \mapsto j-1$ in the left sum and split the right one in two, one in which $j=0$ and one with $j \geq 1$ where we use Proposition 5.1:

$$
\begin{aligned}
&= \sum_{p=1}^{n} \sum_{j=1}^{p+1} e^\theta f_{p,j-1}^{n}(\theta) E^{p+1-j} D^j + \sum_{p=1}^{n} f_{p,0}^{n}(\theta) E^{p+1} \\
&+ \sum_{p=1}^{n} \sum_{j=1}^{p} f_{p,j}^{n}(\theta) E^{p-j} \big(\sum_{k=1}^{j} D^k + E\big),
\end{aligned}
$$

changing in the first line, similarly, the index $p \mapsto p-1$ and expanding the second line

$$
\begin{aligned}
&= \sum_{p=2}^{n+1} \sum_{j=1}^{p} e^\theta f_{p-1,j-1}^{n}(\theta) E^{p-j} D^j + \sum_{p=2}^{n+1} f_{p-1,0}^{n}(\theta) E^{p} \\
&+ \sum_{p=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{j} f_{p,j}^{n}(\theta) E^{p-j} D^k + \sum_{p=1}^{n} \sum_{j=1}^{p} f_{p,j}^{n}(\theta) E^{p+1-j}.
\end{aligned}
$$

In the first term of the second line do the triple change of variables: $p \mapsto p+k, j \mapsto j+k, k \mapsto j$ and in the second term of the second line the double change of variables: $p \mapsto k, j \mapsto k-p+1$.

$$
\begin{aligned}
&= \sum_{p=2}^{n+1} \sum_{j=1}^{p} e^\theta f_{p-1,j-1}^{n}(\theta) E^{p-j} D^j + \sum_{p=2}^{n+1} f_{p-1,0}^{n}(\theta) E^{p} \\
&+ \sum_{p=1}^{n} \sum_{j=1}^{p} \sum_{k=0}^{n-p} f_{p+k,j+k}^{n}(\theta) E^{p-j} D^j + \sum_{p=1}^{n} \sum_{k=p}^{n} f_{k,k-p+1}^{n}(\theta) E^{p}.
\end{aligned}
$$

Finally, grouping the terms by powers of $E^{p-j}D^j$

$$
\begin{aligned}
&= \sum_{k=1}^{n} f_{k,k}^n(\theta)E + \sum_{k=2}^{n} f_{k,k}^n(\theta)D \\
&+ \sum_{p=2}^{n} \left[f_{p-1,0}^n(\theta) + \sum_{k=p}^{n} f_{k,k-p+1}^n(\theta) \right] E^p \\
&+ \sum_{p=2}^{n} \sum_{j=1}^{p} \left[e^\theta f_{p-1,j-1}^n(\theta) + \sum_{k=0}^{n-p} f_{p+k,j+k}^n(\theta) \right] E^{p-j}D^j \\
&+ f_{n,0}^n(\theta)E^{n+1} + \sum_{j=1}^{n+1} e^\theta f_{n,j-1}^n(\theta)E^{n+1-j}D^j.
\end{aligned}
$$

In this last expression we have precisely the form $\sum_{p=1}^{n+1}\sum_{j=0}^{p} f_{p,j}^{n+1}(\theta)E^{p-j}D^j$ from which Equation (5.2) can be read. The functions $f_{p,j}^n$ are all polynomials in e^θ because this holds for the induction hypothesis and the operations in the induction step are only multiplications and additions of polynomials with positive coefficients.

An analogous calculation with $(e^\theta D + E)^{n+1} = (e^\theta D + E)(e^\theta D + E)^n$ will give Equation (5.3). ■

We now state an auxiliary result.

Lemma 5.3. *For $n \geq 2$ and $1 \leq p \leq r \leq n-1$, we have the identity*

$$
\sum_{k=p}^{r} k \binom{n-1-k}{r-k} = \frac{np - pr + r}{n-r+1} \binom{n-p}{r-p}. \tag{5.4}
$$

Proof. First note that the cases $p = r$ and $r = n-1$ are easy to check directly. We prove the general case by induction over n. The case $n = 2$ is again easy to see. We now assume that (5.4) holds for fixed $n \geq 2$ and all $1 \leq p \leq r \leq n-1$.

To show the result for $n+1$ we may assume $1 \leq p < r \leq n-1$, ignoring the easy cases settled at the beginning of the proof. Starting from the left hand side for $n+1$

and using the induction hypothesis on the third equality,

$$
\begin{aligned}
\sum_{k=p}^{r} k\binom{n-k}{r-k} &= \sum_{k=p}^{r-1} k\left[\binom{n-k-1}{r-k-1}+\binom{n-k-1}{r-k}\right]+r \\
&= \sum_{k=p}^{r-1} k\binom{n-k-1}{r-k-1}+\sum_{k=p}^{r} k\binom{n-k-1}{r-k} \\
&= \frac{np-p(r-1)+(r-1)}{n-(r-1)+1}\binom{n-p}{(r-1)-p}+\frac{np-pr+r}{n-r+1}\binom{n-p}{r-p} \\
&= \frac{(n+1)p-pr+r}{(n+1)-r+1}\binom{(n+1)-p}{r-p}.
\end{aligned}
$$

∎

We now identify the coefficients $f^n_{p,p}(\theta)$, for $1 \le p \le n$.

Proposition 5.4. *For the coefficients defined in Proposition 5.2,*

$$
f^n_{p,p}(\theta) = \begin{cases} \displaystyle\sum_{r=p}^{n-1} \frac{p}{n}\binom{n-p-1}{r-p}\binom{n}{r} e^{r\theta} & \text{if } 1 \le p < n, \\ e^{n\theta} & \text{if } p = n. \end{cases} \tag{5.5}
$$

Proof. Putting $j = p$ in equation (5.2) of Proposition 5.2 we get a simplified recursion:

$$
f^1_{1,1}(\theta) = e^{\theta}
$$

and

$$
f^n_{p,p}(\theta) = \begin{cases} \displaystyle\sum_{k=1}^{n-1} f^{n-1}_{k,k}(\theta) & \text{if } n>1,\quad p=1 \\ \displaystyle e^{\theta} f^{n-1}_{p-1,p-1}(\theta) + \sum_{k=p}^{n-1} f^{n-1}_{k,k}(\theta) & \text{if } n>1,\quad 1<p<n \\ e^{\theta} f^{n-1}_{n-1,n-1}(\theta) & \text{if } n>1,\quad p=n. \end{cases} \tag{5.6}
$$

If $p = n$, it is easy to see by induction that

$$
f^n_{n,n}(\theta) = e^{n\theta}. \tag{5.7}
$$

Now, if $p < n$, we proceed again by induction. Here the base of induction has to be $n = 2$. The recursion equation (5.6) gives $f^2_{1,1}(\theta) = f^1_{1,1}(\theta) = e^\theta$, as required by formula (5.5). We now assume that (5.5) holds for fixed $n \geq 2$ and all $p < n$. We first consider the branch of (5.6), referring to the case $p = 1$. Using (5.7) and the induction hypothesis we obtain

$$\begin{aligned} f^{n+1}_{1,1}(\theta) &= \sum_{k=1}^{n} f^n_{k,k}(\theta) = \sum_{k=1}^{n-1} f^n_{k,k}(\theta) + f^n_{n,n}(\theta) \\ &= \sum_{k=1}^{n-1}\sum_{r=k}^{n-1} \frac{k}{n}\binom{n-k-1}{r-k}\binom{n}{r} e^{r\theta} + e^{n\theta}, \end{aligned}$$

changing the order of summation and using Lemma 5.3 gives

$$\begin{aligned} &= \sum_{r=1}^{n-1}\sum_{k=1}^{r} \frac{k}{n}\binom{n-k-1}{r-k}\binom{n}{r} e^{r\theta} + e^{n\theta} \\ &= \sum_{r=1}^{n-1} \frac{1}{n}\binom{n}{r} e^{r\theta} \sum_{k=1}^{r} k\binom{n-k-1}{r-k} + e^{n\theta} \\ &= \sum_{r=1}^{n-1} \frac{1}{n}\binom{n}{r} e^{r\theta} \frac{n}{n-r+1}\binom{n-1}{r-1} + e^{n\theta}, \end{aligned}$$

which we rewrite as

$$= \sum_{r=1}^{n} \frac{1}{n+1}\binom{n-1}{r-1}\binom{n+1}{r} e^{r\theta}.$$

Since this is the result required by the induction step, the case $p = 1$ is settled. We can therefore turn our attention to the remaining branch of (5.6), covering $1 < p < n$. We obtain from the induction hypothesis

$$
\begin{aligned}
f_{p,p}^{n+1}(\theta) &= e^{\theta} f_{p-1,p-1}^{n}(\theta) + \sum_{k=p}^{n} f_{k,k}^{n}(\theta) \\
&= e^{\theta} \sum_{r=p-1}^{n-1} \frac{p-1}{n} \binom{n-p}{r-p+1} \binom{n}{r} e^{r\theta} + \sum_{k=p}^{n-1} \sum_{r=k}^{n-1} \frac{k}{n} \binom{n-k-1}{r-k} \binom{n}{r} e^{r\theta} + e^{n\theta};
\end{aligned}
$$

changing the summation order and grouping the terms by powers of e^{θ} yields

$$
\begin{aligned}
&= \sum_{r=p}^{n} \frac{p-1}{n} \binom{n-p}{r-p} \binom{n}{r-1} e^{r\theta} + \sum_{r=p}^{n-1} \sum_{k=p}^{r} \frac{k}{n} \binom{n-k-1}{r-k} \binom{n}{r} e^{r\theta} + e^{n\theta} \\
&= \sum_{r=p}^{n-1} \left[\frac{p-1}{n} \binom{n-p}{r-p} \binom{n}{r-1} + \sum_{k=p}^{r} \frac{k}{n} \binom{n-k-1}{r-k} \binom{n}{r} \right] e^{r\theta} \\
&\quad + \left[\frac{p-1}{n} \binom{n-p}{n-p} \binom{n}{n-1} + 1 \right] e^{n\theta};
\end{aligned}
$$

simplifying and using Lemma 5.3 in the second line gives

$$
\begin{aligned}
&= \sum_{r=p}^{n-1} \left[\frac{r(p-1)}{n-r+1} \binom{n-p}{r-p} + \sum_{k=p}^{r} k \binom{n-k-1}{r-k} \right] \frac{1}{n} \binom{n}{r} e^{r\theta} + p e^{n\theta} \\
&= \sum_{r=p}^{n-1} \left[\frac{r(p-1)}{n-r+1} \binom{n-p}{r-p} + \frac{np-pr+r}{n-r+1} \binom{n-p}{r-p} \right] \frac{1}{n} \binom{n}{r} e^{r\theta} + p e^{n\theta} \\
&= \sum_{r=p}^{n-1} \left[\frac{r(p-1)}{n-r+1} + \frac{np-pr+r}{n-r+1} \right] \frac{1}{n} \binom{n-p}{r-p} \binom{n}{r} e^{r\theta} + p e^{n\theta};
\end{aligned}
$$

grouping that last term with the rest of the terms in the sum finally results in

$$
= \sum_{r=p}^{n} \frac{p}{n+1} \binom{n-p}{r-p} \binom{n+1}{r} e^{r\theta}.
$$

■

Proposition 5.5. *For all $n \in \mathbb{N}$, $1 \leq p \leq n$ and $0 \leq j \leq p$ we have the symmetry*

$$f^n_{p,j}(\theta) = e^{n\theta} f^n_{p,p-j}(-\theta).$$

Proof. Defining the polynomials $g^n_{p,j}(\theta) := e^{n\theta} f^n_{p,p-j}(-\theta)$, we can write

$$f^n_{p,j}(\theta) = e^{n\theta} g^n_{p,p-j}(-\theta),$$

and by the definition (5.1) of the polynomials $f^n_{p,j}(\theta)$ we obtain by changing the summation index

$$\begin{aligned}(e^\theta D + E)^n &= \sum_{p=1}^{n} \sum_{j=0}^{p} f^n_{p,j}(\theta) E^{p-j} D^j = \sum_{p=1}^{n} \sum_{j=0}^{p} e^{n\theta} g^n_{p,p-j}(-\theta) E^{p-j} D^j \\ &= \sum_{p=1}^{n} \sum_{j=0}^{p} e^{n\theta} g^n_{p,j}(-\theta) E^j D^{p-j}. \end{aligned} \tag{5.8}$$

Evaluating this expression for $n = 1$, we find

$$e^\theta D + E = e^\theta g^1_{1,0}(-\theta) D + e^\theta g^1_{1,1}(-\theta) E,$$

and hence $g^1_{1,1}(\theta) = e^\theta$ and $g^1_{1,0}(\theta) = 1$. Next we find a recursive relation for these polynomials by expanding and employing (5.8),

$$\begin{aligned}(e^\theta D + E)^{n+1} &= (e^\theta D + E)^n (e^\theta D + E) \\ &= \left(\sum_{p=1}^{n} \sum_{j=0}^{p} e^{n\theta} g^n_{p,j}(-\theta) E^j D^{p-j} \right) (e^\theta D + E)\end{aligned}$$

and equating the coefficients to

$$(e^\theta D + E)^{n+1} = \sum_{p=1}^{n+1} \sum_{j=0}^{p} e^{(n+1)\theta} g^{n+1}_{p,j}(-\theta) E^j D^{p-j}$$

we find that the polynomials $g^n_{p,j}$ satisfy the following recursion:

$$g^1_{1,0}(\theta) = 1, \qquad g^1_{1,1}(\theta) = e^\theta$$

$$g^n_{p,j}(\theta) = \begin{cases} \displaystyle\sum_{k=1}^{n-1} e^\theta g^{n-1}_{k,0}(\theta) & \text{if } n > 1, \quad p = 1, \qquad j \in \{0,1\} \\ \displaystyle g^{n-1}_{p-1,j}(\theta) + \sum_{k=p}^{n-1} e^\theta g^{n-1}_{k,j}(\theta) & \text{if } n > 1, \quad 1 < p < n, \quad 0 \le j < p \\ \displaystyle e^\theta g^{n-1}_{p-1,p-1}(\theta) + \sum_{k=p}^{n-1} e^\theta g^{n-1}_{k,p-1}(\theta) & \text{if } n > 1, \quad 1 < p < n, \quad j = p \\ g^{n-1}_{n-1,j}(\theta) & \text{if } n > 1, \quad p = n, \qquad 0 \le j < n \\ e^\theta g^{n-1}_{n-1,n-1}(\theta) & \text{if } n > 1, \quad p = n, \qquad j = n \end{cases}$$

This is the recursion equation (5.3) of Proposition 5.2, and hence $f^n_{p,j}(\theta) = g^n_{p,j}(\theta)$. Thus from the definition of $g^n_{p,j}(\theta)$ we conclude that $f^n_{p,j}(\theta) = e^{n\theta} f^n_{p,p-j}(-\theta)$ as claimed. ∎

From the symmetry rule of Proposition 5.5 we obtain a simple characterisation of the coefficients $f^n_{p,0}(\theta)$.

Corollary 5.6.

$$f^n_{p,0}(\theta) = \begin{cases} \displaystyle\sum_{r=1}^{n-p} \frac{p}{n} \binom{n-p-1}{r-1} \binom{n}{r} e^{r\theta} & \textit{if } 1 \le p < n \\ 1 & \textit{if } p = n. \end{cases}$$

Proof. The result follows by combining Proposition 5.5 for $j = p$, and Proposition 5.4. ∎

5.2 A couple of lower bounds

With all of these combinatorial results we come back to our problem at hand. We now plug the expansion (5.1) into equation (3.21) and obtain

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{n} \log \left\{ \sum_{p=1}^{n} \sum_{j=0}^{p} f_{p,j}^{n}(\theta) w^T E^{p-j} D^j v \right\} + \log c. \tag{5.9}$$

Since all terms are positive, we can find lower bounds by considering only the terms for which $j = 0$ or $j = p$. This is the content of the next couple of results.

Proposition 5.7. *For the cumulant generating function* Λ *defined by* (3.21),

$$\Lambda(\theta) \geq \begin{cases} 2\log\left(1 + e^{\theta/2}\right) + \log c & \text{if } \theta \leq -2\log\lambda_1, \\ \log\left(1 + \lambda_1 e^{\theta}\right) + \log\left(1 + \dfrac{1}{\lambda_1}\right) + \log c & \text{if } \theta > -2\log\lambda_1. \end{cases}$$

Proof. From the explicit form of D, w, and v in (3.6) it can be shown by induction that

$$w^T D^p v = (1+\lambda_1)^p \frac{\alpha}{\lambda_1 - \lambda_2} \left\{ \frac{\lambda_1}{\alpha - \lambda_1(1-\alpha)} - \left(\frac{1+\lambda_2}{1+\lambda_1}\right)^p \frac{\lambda_2}{\alpha - \lambda_2(1-\alpha)} \right\}. \tag{5.10}$$

Note that for region *(b)* of Theorem 2.3, we have that $\lambda_1 = \lambda_2 = 1$ and the previous expression is undefined. However, taking the limit we find

$$\lim_{c\to\frac{1}{4}} w^T D^p v = 2^p \left\{ \left(\frac{\alpha}{2\alpha - 1}\right)^2 + \frac{p}{2}\frac{\alpha}{2\alpha - 1} \right\},$$

and we can continue our calculations with the general case, so considering only the terms for which $j = p$ in equation (5.9) and note that the expression in braces in (5.10) vanishes in the limit taken in (5.9). Hence, using the Laplace principle and Proposition 5.4,

$$\begin{aligned}\Lambda(\theta) &\geq \lim_{n\to\infty}\frac{1}{n}\log\left\{\sum_{p=1}^{n} f_{p,p}^{n}(\theta)(1+\lambda_1)^p\right\}+\log c\\ &= \lim_{n\to\infty}\sup_{1\leq p\leq r\leq n}\frac{1}{n}\log\left\{\frac{p}{n}\binom{n-p-1}{r-p}\binom{n}{r}e^{r\theta}(1+\lambda_1)^p\right\}+\log c.\end{aligned}$$

We now use a change of variables $\varepsilon=\dfrac{r}{n}$ and $\delta=\dfrac{p}{n}$ and use Stirling's Formula to obtain

$$\begin{aligned}\Lambda(\theta) &\geq \sup_{0<\delta\leq\varepsilon\leq 1}\left[\lim_{n\to\infty}\frac{1}{n}\log\left\{\delta\binom{n(1-\delta)}{n(\varepsilon-\delta)}\binom{n}{n\varepsilon}\right\}+\varepsilon\theta+\delta\log(1+\lambda_1)\right]+\log c\\ &= \sup_{0<\varepsilon\leq 1}\left[\sup_{0<\delta\leq\varepsilon}\left\{-(\varepsilon-\delta)\log(\varepsilon-\delta)+(1-\delta)\log(1-\delta)+\delta\log(1+\lambda_1)\right\}\right.\\ &\qquad\left.+\,\varepsilon\theta-2(1-\varepsilon)\log(1-\varepsilon)-\varepsilon\log\varepsilon\right]+\log c.\end{aligned}$$

The inner problem, when ε is fixed, is solved by choosing

$$\delta^{\max}=\begin{cases}0 & \text{if } \varepsilon\leq\dfrac{1}{1+\lambda_1},\\[2ex] \dfrac{(1+\lambda_1)\varepsilon-1}{\lambda_1} & \text{if } \varepsilon>\dfrac{1}{1+\lambda_1}.\end{cases}$$

So now we have

$$\begin{aligned}\Lambda(\theta) \quad\geq\quad \max\Bigg\{ &\sup_{0<\varepsilon\leq\frac{1}{1+\lambda_1}}\left[\varepsilon\theta-2\varepsilon\log\varepsilon-2(1-\varepsilon)\log(1-\varepsilon)\right],\\ &\sup_{\frac{1}{1+\lambda_1}<\varepsilon\leq 1}\left[\varepsilon\theta-\varepsilon\log\varepsilon-(1-\varepsilon)\log[\lambda_1(1-\varepsilon)]+\log(1+\lambda_1)\right]\Bigg\}+\log c.\end{aligned}$$

This problem is solved by choosing

$$\varepsilon^{\max} = \begin{cases} \dfrac{e^{\theta/2}}{1+e^{\theta/2}} & \text{if } \theta \le -2\log\lambda_1 \\[2em] \dfrac{\lambda_1 e^{\theta}}{1+\lambda_1 e^{\theta}} & \text{if } \theta > -2\log\lambda_1. \end{cases}$$

Plugging this value of $\varepsilon^{\max}$ yields the result of the proposition. ■

Proposition 5.8. *For the cumulant generating function* Λ *defined by* (3.21)*,*

$$\Lambda(\theta) \ge \begin{cases} \log\left(\dfrac{e^{\theta}}{1-\alpha} + \dfrac{1}{\alpha}\right) + \log c & \text{if } \theta \le 2\log\left(\dfrac{1}{\alpha} - 1\right), \\ 2\log\left(1+e^{\theta/2}\right) + \log c & \text{if } \theta > 2\log\left(\dfrac{1}{\alpha} - 1\right). \end{cases}$$

Proof. We follow the same technique as in the previous proposition, once again from the explicit form of E, w, and v in (2-5) and (3.1b) we have

$$w^T E^p v = \frac{1}{\alpha^p} w^T v,$$

and recall we calculated an explicit expression for $w^T v$ in (3.20).

Now consider only those values for which $j = 0$ in (5.1). Hence, using Corollary 5.6

$$\begin{aligned} \Lambda(\theta) &\ge \lim_{n\to\infty} \frac{1}{n} \log\left\{ \sum_{p=1}^{n} f^n_{p,0}(\theta) \frac{1}{\alpha^p} \right\} + \log c \\ &= \lim_{n\to\infty} \sup_{\substack{1\le p\le n \\ 0\le r\le n-p}} \frac{1}{n} \log\left\{ \frac{p}{n} \binom{n-p-1}{r-1} \binom{n}{r} e^{r\theta} \alpha^{-p} \right\} + \log c \end{aligned}$$

With the same change of variables as before, $\varepsilon = \frac{r}{n}$ and $\delta = \frac{p}{n}$, we have

$$\begin{aligned}
\Lambda(\theta) &\geq \sup_{\substack{0<\delta\leq 1\\ 0\leq\varepsilon\leq 1-\delta}} \left[\lim_{n\to\infty}\frac{1}{n}\log\left\{\delta\binom{n(1-\delta)}{n\varepsilon}\binom{n}{n\varepsilon}\right\}+\varepsilon\theta-\delta\log\alpha\right]+\log v\\
&= \sup_{\substack{0\leq\delta\leq 1\\ 0\leq\varepsilon\leq 1-\delta}} \{(1-\delta)\log(1-\delta)-(1-\delta-\varepsilon)\log(1-\delta-\varepsilon)-\delta\log\alpha\\
&\qquad -2\varepsilon\log\varepsilon-(1-\varepsilon)\log(1-\varepsilon)+\varepsilon\theta\}+\log c.
\end{aligned}$$

Splitting the problem in two, for a fixed ε we find the optimal δ by

$$\delta^{\max} = \begin{cases} 0 & \text{if } 1-\alpha\leq\varepsilon\leq 1,\\ 1-\dfrac{\varepsilon}{1-\alpha} & \text{if } 0\leq\varepsilon<1-\alpha. \end{cases}$$

And the remaining problem is solved by choosing the optimum ε as

$$\varepsilon^{\max} = \begin{cases} \dfrac{e^{\theta/2}}{1+e^{\theta/2}} & \text{if } \theta\leq 2\log\left(\dfrac{1}{\alpha}-1\right),\\ \dfrac{\alpha}{\alpha+e^{-\theta}(1-\alpha)} & \text{if } \theta> 2\log\left(\dfrac{1}{\alpha}-1\right). \end{cases}$$

With these values of ε we get the desired result. ■

Corollary 5.9. *For* Λ *defined by* (3.21),

$$\Lambda(\theta)\geq\begin{cases} \log\left(\dfrac{e^{\theta}}{1-\alpha}+\dfrac{1}{\alpha}\right)+\log c & \textit{if } -\infty<\theta\leq 2\log\left(\dfrac{1}{\alpha}-1\right),\\ 2\log\left(1+e^{\theta/2}\right)+\log c & \textit{if } 2\log\left(\dfrac{1}{\alpha}-1\right)<\theta\leq -2\log\lambda_1,\\ \log\left(1+\lambda_1 e^{\theta}\right)+\log\left(1+\dfrac{1}{\lambda_1}\right)+\log c & \textit{if } -2\log\lambda_1<\theta<\infty. \end{cases}$$

Proof. The bounds from Propositions 5.7 and 5.8 are the same in the interval $2\log\left(\frac{1}{\alpha}-1\right)\leq\theta\leq -2\log\lambda_1$. In the other intervals, a comparison of the bounds establishes the claim. ■

5.3 The Legendre Transform

Summarising, we have the following result.

Corollary 5.10. *For the cumulant generating function* Λ *defined by* (3.21),

$$\Lambda(\theta) = \begin{cases} \log\left(\dfrac{e^\theta}{1-\alpha}+\dfrac{1}{\alpha}\right)+\log c & \text{if } -\infty<\theta\le 2\log\left(\dfrac{1}{\alpha}-1\right), \\ 2\log\left(1+e^{\theta/2}\right)+\log c & \text{if } 2\log\left(\dfrac{1}{\alpha}-1\right)<\theta\le -2\log\lambda_1, \\ \log\left(1+\lambda_1 e^\theta\right)+\log\left(1+\dfrac{1}{\lambda_1}\right)+\log c & \text{if } -2\log\lambda_1<\theta<\infty. \end{cases} \tag{5.11}$$

Proof. This follows from the fact that the upper and lower bounds from Proposition 4.7 and Corollary 5.9, respectively, are the same. ∎

Finally we have the necessary tools to prove Theorem 2.3.

Proof. The rate function in the case $\alpha \le 1/2$ and $\rho \le 1-\alpha$ is known from Cramér's Theorem, see e.g. Exercise 2.2.23 (b) in [10], alternatively we proved it using Gärtner-Ellis Theorem on (3.22) giving the rate function (3.23).

For the case $\alpha > 1/2$ and $\frac{1}{2} < \rho < \alpha$, we show that the function Λ defined by (3.21), given explicitly in Corollary 5.10, satisfies the hypotheses of the Gärtner-Ellis Theorem 3.6. Note that Λ is defined for all real numbers. An evaluation at the boundaries of the domains gives

$$\Lambda\left(2\log(\tfrac{1}{\alpha}-1)\right) = -2\log\alpha+\log c = \lim_{h\to 0^+}\Lambda\left(2\log(\tfrac{1}{\alpha}-1)-h\right)$$

and

$$\Lambda\left(-2\log\lambda_1\right) = 2\log(1+\tfrac{1}{\lambda_1})+\log c = \lim_{h\to 0^+}\Lambda\left(-2\log\lambda_1+h\right),$$

which implies that it is continuous in all $\mathbb{R}$. Moreover,

$$\lim_{h\to 0^\pm}\frac{\Lambda\left(2\log(\frac{1}{\alpha}-1)+h\right)-\Lambda\left(2\log(\frac{1}{\alpha}-1)\right)}{h} = 1-\alpha$$

and

$$\lim_{h\to 0^\pm} \frac{\Lambda\left(-2\log\lambda_1 + h\right) - \Lambda\left(-2\log\lambda_1\right)}{h} = \frac{1}{1+\lambda_1}.$$

Therefore, Λ is differentiable in $\mathbb{R}$. By the Gärtner-Ellis Theorem we just have to find its Legendre transform to find the rate function,

$$I(x) = \sup_{\theta\in\mathbb{R}} \{x\theta - \Lambda(\theta)\} \qquad \text{for } x \in (0,1).$$

For fixed $x \in (0,1)$ the function $\theta \mapsto x\theta - \Lambda(\theta)$ is defined, continuous, and differentiable in all $\mathbb{R}$. It is also a concave function and hence the maximum is reached at a value of θ where the derivative vanishes. Since

$$\frac{d}{d\theta}\left(x\theta - \Lambda(\theta)\right) = \begin{cases} x - \dfrac{\alpha e^\theta}{\alpha e^\theta + 1 - \alpha} & \text{if } -\infty < \theta \le \log\left(\dfrac{1}{\alpha} - 1\right)^2, \\ x - \dfrac{e^{\theta/2}}{1+e^{\theta/2}} & \text{if } \log\left(\dfrac{1}{\alpha} - 1\right)^2 < \theta \le -2\log\lambda_1, \\ x - \dfrac{\lambda_1 e^\theta}{1+\lambda_1 e^\theta} & \text{if } -2\log\lambda_1 < \theta < \infty, \end{cases}$$

we get $\dfrac{d}{d\theta}\left(x\theta - \Lambda(\theta)\right) = 0$ if and only if

$$\begin{aligned} &\theta = \log\frac{x(1-\alpha)}{\alpha(1-x)} && \text{and} \quad \theta \le 2\log\left(\frac{1}{\alpha} - 1\right), \\ \text{or} \quad &\theta = 2\log\frac{x}{1-x} && \text{and} \quad 2\log\left(\frac{1}{\alpha} - 1\right) < \theta \le -2\log\lambda_1, \\ \text{or} \quad &\theta = \log\frac{x}{\lambda_1(1-x)} && \text{and} \quad \theta > -2\log\lambda_1. \end{aligned}$$

This means that

$$\begin{aligned} \theta &= \log\frac{x(1-\alpha)}{\alpha(1-x)} &&\Leftrightarrow \quad 0 < x < 1-\alpha, \\ \theta &= 2\log\frac{x}{1-x} &&\Leftrightarrow \quad 1-\alpha < x \le \frac{1}{1+\lambda_1}, \\ \theta &= \log\frac{x}{\lambda_1(1-x)} &&\Leftrightarrow \quad \frac{1}{1+\lambda_1} < x < 1. \end{aligned}$$

Since the value of θ that satisfies $\frac{d}{d\theta}(x\theta - \Lambda(\theta)) = 0$ is unique it must be the maximum. By plugging in this value in (5.11), we reach

$$I(x) = \begin{cases} x \log \dfrac{x}{\alpha} + (1-x)\log \dfrac{1-x}{1-\alpha} + \log \dfrac{\alpha(1-\alpha)}{c} & \text{if} \quad 0 \le x \le 1-\alpha, \\ 2\left[x \log x + (1-x)\log(1-x) - \log\sqrt{c}\right] & \text{if} \quad 1-\alpha < x \le \dfrac{1}{1+\lambda_1}, \\ x \log x + (1-x)\log[\lambda_1(1-x)] - \log[c(1+\lambda_1)] & \text{if} \quad \dfrac{1}{1+\lambda_1} < x \le 1. \end{cases}$$

We reach the desired result substituting $c = \rho(1-\rho)$ and $\lambda_1 = \dfrac{\rho}{1-\rho}$.

For the case $\alpha > 1/2$ and $\rho \le \frac{1}{2}$, we just evaluate the previous case at $c = \frac{1}{4}$ which implies $\lambda_1 = 1$. This finishes the proof of Theorem 2.3.

■

CHAPTER 6

FINAL REMARKS

6.1 Hydrodynamic limits

Somewhat selfishly, one of the reasons to focus on the study of the types of problems discussed in this thesis is that there is an overlap of analysis and probability techniques while at the same time having a physical interpretation. The concept of hydrodynamic limit is a beautiful example of this.

One (very romantic) way of thinking of a hydrodynamic limit is as follows. Imagine you see a system from up close that consists of particles moving stochastically on a finite lattice, as an example think of a drop of sea water and all its molecules seen from a microscope. Since we are very close to the system we are only able to see a small amount of particles and sites and so we take a step farther: we know see more, but we see them smaller. As we walk farther, we see more particles on a larger lattice on our same visual frame. Eventually, we will not be able to distinguish between sites nor particles seeing only a continuum which somehow does not have a stochastic behaviour any more, as a wave hitting the shore on the beach.

Formally, as an example, we will consider the finite ASEP $\{\xi(t)\}_{t\geq 0}$ on the torus with n sites with rate of a particle jumping to the right given by $p \in (0,1) \setminus \{\frac{1}{2}\}$ and $1-p$ to the left. That is, it has the state space $\{0,1\}^{\mathbb{T}_n}$ and generator

$$\begin{aligned} Gf(\eta) = &\sum_{k\in\mathbb{T}_n} p\eta_k(1-\eta_{k+1})\left(f(\sigma^{k,k+1}\eta) - f(\eta)\right) \\ &+ \sum_{k\in\mathbb{T}_n} (1-p)\eta_k(1-\eta_{k-1})\left(f(\sigma^{k,k-1}\eta) - f(\eta)\right). \end{aligned}$$

Denote by $\mathbb{T}$ the one dimensional torus. Let $u_0 \colon \mathbb{T} \to [0,1]$ and define the associated initial product measure μ_0 on $\mathbb{T}_n$ by

$$\mu_0[\eta : \eta_k = 1] = u_0(k/n)$$

Note that, by construction, for all $\varepsilon > 0$ and smooth functions with compact support $\varphi \colon \mathbb{T} \to \mathbb{R}$

$$\lim_{n\to\infty} \mu_0^n \left[\left| \frac{1}{n} \sum_{k\in\mathbb{T}_n} \eta_k \varphi(\tfrac{k}{n}) - \int_{\mathbb{T}} \varphi(x) u_0(x) dx \right| > \varepsilon \right] = 0.$$

Denote by μ_t^n the evolution of μ_0^n, that is, if $S(t)$ is the semigroup then $\mu_t^n = \mu_0^n S(t)$. We then say that the function $u \colon \mathbb{T} \times [0,T] \to [0,1]$ is the density profile of the hydrodynamic limit if for all $\varepsilon > 0$ and smooth functions with compact support $\varphi \colon \mathbb{T} \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \mu_{nt}^n \left[\left| \frac{1}{n} \sum_{k\in\mathbb{T}_n} \eta_k \varphi(\tfrac{k}{n}) - \int_{\mathbb{T}} \varphi(x) u(x,t) dx \right| > \varepsilon \right] = 0.$$

Benassi [3] showed that for the ASEP the density profile is the solution to Burgers's equation,

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial}{\partial x} u(1-u) = 0, \qquad u(x,0) = u_0(x).$$

In the real line, Burgers's equation may generate shocks, see figure 6-1, or rarefaction fans, see figure 6-2, depending on the initial conditions.

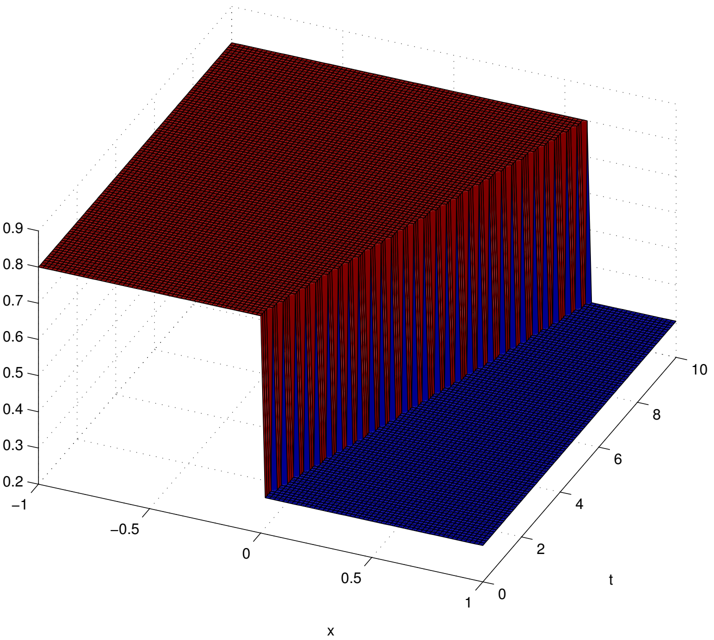

Figure 6-1: Shock solution to Burgers's equation

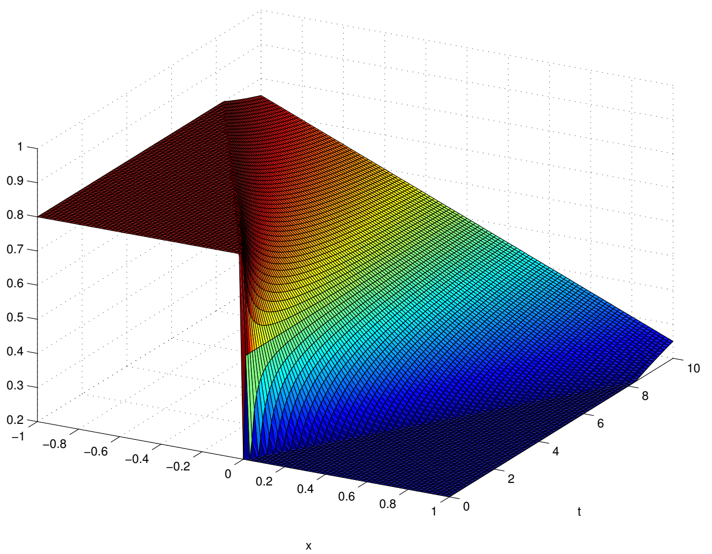

Figure 6-2: Rarefaction fan solution to Burgers's equation.

So, basically, a hydrodynamic limit is a statement of the weak law of large numbers. This is why it is natural to ask about large deviations. However, our main result, Theorem 2.3, does not involve the whole density profile but rather an interval close to the boundary. The large deviation principle for the semi-infinite TASEP is still an open problem and this comment brings us to the next section.

6.2 Open problems

Solving our original problem only gives birth to even more questions. In this section we will show just a few of the questions that follow quite naturally from our result.

6.2.1 LDP for the complete parameter space

Our main result, Theorem 2.3, does not address the region where $1-\rho \leq \alpha \leq \rho$ which corresponds to regions 2 and 3 from figure 3-2. The next step that we think should be considered for further research is precisely to complete this picture.

6.2.2 Dynamical large deviations

Many physical systems are far from equilibrium and so we need to understand their behaviour dynamically rather than under stationarity. It would be interesting to find a time dependent large deviation rate functional for an interacting particle system which will give us a better understanding of the system.

In contrast with what we know for the boundary driven semi-infinite TASEP under stationarity, we now let it depend on time $\{\xi(t)\}_{t\geq 0}$ and study the time dependent empirical density

$$X_n(t) = \frac{1}{n}\sum_{k=1}^{n} \xi_k(nt).$$

The objective is to obtain for a fixed $t \geq 0$ a rate function $I_t\colon [0,1] \to [0,\infty]$ and assess whether taking the limit as $t \to \infty$ will converge to the rate function of the process under stationarity.

This approach takes advantage of our method developed for the system under stationarity and the work of Stinchcombe and Schütz [26] that proposes a framework of time dependent matrix products.

Moreover, we can do even better if we study the empirical density as a measure, that is

$$\mu_n(t) = \frac{1}{n} \sum_{k \in \mathbb{N}} \xi_k(nt) \delta_{k/n}.$$

Finding a time dependent rate functional acting on the space of probability measures of the half line $J_t \colon \mathcal{M}[0, \infty) \to [0, \infty]$ that is related with our main result via the contraction principle

$$I_t(x) = \inf\{J_t(\mu) : \mu[0, 1] = x\}.$$

6.2.3 Processes with MPA description

Besides the TASEP, which has served as a paradigm of interacting particle systems, there exist many other physically relevant systems. Adding complexity in the right direction to previously studied models can ultimately impact on practical applications. For example, a challenging question is to understand processes where particles have an exchange of mass or energy when an interaction happens.

Consider a model in which interaction of particles radically change their stochastic behaviour. Specifically, one in which particles move in the same direction until they hit a particle moving in the opposite way. At this moment, each of the particles involved in the collision changes direction.

Formally, the process $\{\xi(t)\}_{t \geq 0}$ has a state space $\Omega = \{-1, 0, 1\}^{\mathbb{Z}}$. The vector $\eta \in \Omega$ is understood as the configuration with value 1 where there is a particle moving at rate 1 to the right, 0 where there is no particle, and -1 where there is a particle moving at rate 1 to the left. The infinitesimal generator G of this object is

$$Gf(\eta) = \sum_{x\in\mathbb{Z}} |\eta_x|(1-|\eta_{x+\eta_x}|)(f(\sigma^{x,x+\eta_x}(\eta)) - f(\eta))$$
$$+ \sum_{x\in\mathbb{Z}} \mathbb{1}_{\{|\eta_x-\eta_{x+\eta_x}|=2\}}(f(\sigma^{x,x+\eta_x}(\eta)) - f(\eta)).$$

Where the function $\sigma^{x,y}$ changes the vector η by switching its components x and y, as before, and $f\colon \Omega \to \mathbb{R}$ is a function that depends only on a finite number of coordinates of η.

Several tasks may be addressed in the analysis of this model: Find the hydrodynamic limit. What if instead of changing direction with certainty, this only occurs with probability p? Find a large deviation principle for this limit? If the model accepts a matrix product representation then we can use our method, otherwise we want to know why it cannot be used and propose a new method that works under these conditions.

6.3 Concluding discussion

There are two contributions of this work: The explicit rate function for the large deviation principle under stationarity of the empirical density of the semi-infinite totally asymmetric simple exclusion process and the method consisting on finding the lower and upper bounds for the limit cumulant moment generating function.

There is hope that the technique followed here may be used to find rate functions of large deviation principles for other spatially correlated distributed systems accepting a matrix product representation. For instance, this method can be applied to the other versions of SEP in finite or infinite lattices too and not only the semi-infinite TASEP. It is will be interesting to find other processes for we could apply this method.

It is fair to say that there may be certain concerns as to whether or not the implementation of the method may be as straight forward as the example we are providing. We can already appreciate differences on how implementations are different simply by comparing the cases $\alpha \leq \frac{1}{2}$ and $\alpha > \frac{1}{2}$.

Our method relies on the explicit form of the matrices and vectors of the MPA; moreover, we are taking advantage of the matrix being a Toeplitz operator to find the upper bound on the limit cumulant moment generating function. Other processes may accept a matrix product representation without forming a Toeplitz operator, in this case finding the spectral radius may become harder than the way we did it here. Or even if an operator is Toeplitz, the symbol might not necessarily be a nice ellipse like the one of the semi-infinite TASEP complicating the calculations to find the spectral radius.

Besides the questions arising with the problem at hand, we would like to mention a couple of questions that might not necessarily have a direct relation with the subject at hand but that may be considered mathematically interesting in other areas and caught our curiosity on our multiple attempts of finding an answer to our problem.

In Proposition 2.1 we claim that the generator of a finite TASEP may be expressed as a matrix with an explicit form. There is then some hope of finding the general vector form of the stationary measure by solving the system $\mu Q = 0$, this appears to be a numerical analysis or linear algebra problem of the sort where the matrix to be considered has many zeroes in its entries. If this can be done then we might find another interpretation of the matrices and vectors from the MPA by comparing the two solutions.

The second problem lies completely in the field of analytic combinatorics. In Proposition 5.2 we proved the existence of the polynomials $f^n_{p,j}(\theta)$. For given n, p, and j what is the degree of this polynomial? Can we find the coefficients explicitly without using the recursion? A comparison with Pascal's triangle is unavoidable, but whereas this is a two dimensional arrangement of the numbers counting in how many ways one can choose k elements from a set of n; our coefficients would likely be arranged in a four dimensional "triangle". What are our coefficients counting? Besides, of course, the number of factors the expansion of $(e^{\theta}D+E)^n$ where D and E satisfy conditions (3.1a).

Our result has been stated and has been proved. We have highlighted the good, the bad, and what remains to be done for there is nothing ugly about mathematics, or as Bertrand Russell once put it:

> *"Mathematics, rightly viewed, possesses not only truth, but supreme beauty..."*

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